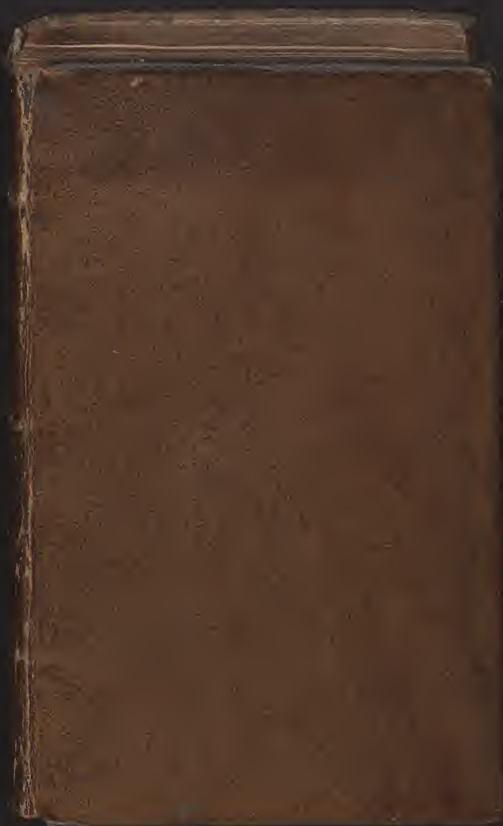




INTRODUCTIO
IN
TETRAGONOMETRIAM
ADMENTEM V. C. L'AMBERT
ANALYTICE CONSCRIPTA

A

STEPHANO BIÖRNSEN

MATHEM. ET PHILOSOPH. CULTORE,



HAVNIAE,
APUD PROFTIUM, UNIVERS. BIBLIOPOL.
MDCCLXXX.

PRINCIPI
HAEREDITARIO,
REGIS FRATRI.

CELSISSIME PRINCEPS
ATQUE SERENISSIME!

Scientiis mathematicis ab omni aevo principes et Magnates magnum pretium ſtatuerunt, easque ſua munificentia promoverunt, cujus rei numeroſa proferri poſſent exempla, majorum parentumque tuorum celſiſſime **Princeps!** *exempla duntaxat commemoro, Friderici II. Tychonis ſummi Aſtronomi Mecenatis, Friderici V. qui pro regia munificentia, qua in rem litterariam uſus eſt, ſumtus ingentes fecit in longinquas expeditiones litterarias, ad Hiſtoriam naturalem, Aſtronomiam, Geographiam et Phyſicam promo-*

 vendas.

vendas. Tu, clarissime Princeps! virtutum heroicarum, quibus aucti fuere majores et parentes tui, imitator dignissimus Historiam civilem, Daniae et Norvegiae nec non antiquitates Septentrionis, mirifico studio et regia liberalitate promovere coepisti, quo ipso in tuae celsitudinis tanto majorem laudem et admirationem homines rapiuntur, quanto minor est aetatis tuae iste modulus, ut ex vero de te praedicandum sit, quod Curtius Rufus quondam de Alexandro magno praedicavit, modulum aetatis omnia ipsius facta heroica honestare. Haec sane gravissima suberat ratio, cur hanc in Tetragonometriam introductionem tuo Numini Nominique potissimum inscribendam censuerim, cujus quidem construendae lapidem angularem jecerat vir magnus Johannes Henricus Lambert, Academiae Berolinensis dum vixit sodalis, ad quem ego caeteros lapides adjeci, ordinavi parietes

rietes construxi, atque totum hoc aedificium Tetragonometricum excitavi; ut asseverare ausim; haud inficias iturum virum Clariss. Lambert, si, quod toto animo exoptarem, adhuc in vivis esset, me quoque se cum participasse, in hac scientia prorsus nova ac utilissima condenda. Sicut enim inde a pueritia in omni studio mathematico assiduus, partem maximam temporis et operam indefessam Mathesi impendi; ita etiam in hoc opere tetragonometrico evolvendo multum temporis consumsi et laboris exantlavi, totius enim Tetragonometriae argumenta evolvi, demonstravi, ordinavi, concinnavi, in systema redegi, et hac ratione totius Polygonometriae, satis vastae, utilis ac inventu difficilis scientiae, haud sane contemnenda fundamina jeci, quos inter labores, certo salario destitutus, pro quotidiano pane comparando multa tuli fecique sudavi et alsi. Nullus igitur du-

 bitavi,

bitavi, celsissime Princeps, hoc opusculum ad pedes tuos humillime deponere, certo persuasus Tibi scientiarum Patrono, hosce meos labores haud improbatum iri. Deus Te servet Principum Daniae decus. Ita vovet.

CELSISSIME PRINCEPS
SERENISSIMI NOMINIS TUI

Havniae
d. XXVIII. Martii
MDCCLXXX.

humillimus cultor
STEPHANUS BIÖRNSEN



PRAEFATIO.

Dari ſyſtema quoddam problematum, quae per Trigonometriam proprie dictam non ſolverentur, ſed quorum objectum eſſent figurae rectilineae quadrilaterae ſive trapezia, et ad claſſem ſibi propriam et peculiarem referrentur, primus omnium Clariſſ. Lambert, Academiae Berolinenſis quondum membrum, animadvertit, et ſcientiae, quae circa iſta problemata occupatur, ex natura rei Tetragonometriae nomen impoſuit. Mirum ſane videri poteſt, Geometras hoc prius non vi-

diſſe. Hujus enim ſcientiae caſus ſimpliciſſimi in vulgaribus Geometriae elementis tradi ſolent hoc problematae: Datis omnibus lateribus figurae cujuscunque rectilineae, et tot diagonalibus, ſive angulis quot ſunt latera, demtis tribus, figuram conſtruere. Quod ſi ergo ad quadrilaterum applicetur problema proponitur hoc modo: Datis omnibus lateribus et angulorum uno, ſive diagonalium altera, conſtruere figuram quadrilateram. Si ad Pentagonum, ita: Datis omnibus lateribus et duobus angulis, vel etiam duabus diagonalibus conſtruere Pentagonum. Si itaque animadvertiſſent Geometrae, neceſſarium haud eſſe, ut omnia dentur latera, ſed uno latere vel pluribus ex quaeſtione exulantibus totidem angulos poſſe ſubſtitui, longe certe prius in hanc

theo-

theoriam incidiſſent. Verum in Tetragonometria recte ſtatuit Clariſſ. Lambert, praeter latera, diagonalem et angulos nullas determinationes adhibendas eſſe, ſicuti in Trigonometria non niſi anguli et latus uſurpantur, et quemadmodum, ſi plures aliae determinationes in triangulis admittantur, Trigonoſcopia vocatur, ita ſiquidem in trapeziorum doctrina plures determinationes admitterentur, Tetragonoſcopia rectius diceretur.

Caeterum Tetragonometriae problema generale, quod omnia ſuo ambitu comprehendit eſt hujusmodi: In figura quadrilatera rectilinea quinque determinationibus datis, ſextam invenire, et conſtruere figuram. Pentagonometriae problema generale foret: In Pentagono quocunque ſeptem determinationibus datis,

tis, octavam invenire et conftruere figuram. Abftrahendo ab ideis fpecierum, *trianguli*, *trapezii*, *pentagoni*, *hexagoni* &c. et ad ideam generis afcendendo, tota hujusmodi fcientia Poligonometria recte vocatur, vafta fatis fcientia, cujus pars maxima adhuc defideratur, primum gradum Trigonometria, fecundum Tetragonometria, tertium Pentagonometria conftituit &c.

Verum uti Trigonometria Tetragonometriae inveniendae principia tradit, ita haec una cum illa Pentagonometriae principia inveniendi tradere debet. Sunt autem problemata Tetragonometriae duorum generum; nam alia utrique methodo, tam Trigonometriae, quam Tetragonometriae fubfunt, alia foli Tetragonometriae propria trigonometrice non folvuntur.

Haec

Haec itaque problemata Tetragonometriae, quorum recenſionem et diviſionem in claſſes et capita dederat primus magnus ille vir Clariſſ. Lambert, methodo analytica breviter expoſui, illi diviſioni et ſubdiviſioni inhaerens, quam auctor ipſe problematum fecerat. Equidem duplo vel etiam triplo prolixius hac de materia ſcribere potuiſſem, ſed brevitati ſtudui, et veritatum delectum habui. Cum enim multae ſint veritates perfaciles, nolui earum multitudine lectori nauſeam movere, et eas ſelegi, quae majoris momenti videbantur, et profecto haud ita paucae veritates interſperguntur, quae attentionem ſeriam et acumen requirunt.

In tota tractatione per ſcholia indicavi caſus problematis Tetragonometriae proprios et

a cae-

a caeteris diſtinctos, quos cum Trigonometria communes habet, illorum utilitatem in Geometria practica indicavi et ſolutiones per aequationes quadraticas ſaepius obtinui, aliquot tamen aequationes ad quartum gradum aſcendunt, ad quas igitur ſtudio brevitatis exemplo Newtoni in Arithmetica univerſali et plurium aliorum reſolutionem problematis deduxiſſe contentus fui, complementum ſolutionis per cognitas methodos ipſi lectori relinquens; habent autem hae aequationes quarti gradus pleraeque formam ſimilem, ita ut unius ſolutione completa data, omnium caeterarum ſolutiones completae habeantur. Pauci dantur caſus in quibus, cum aequationem ad gradum quarto altiorem aſcendere et inſuper magna terminorum multitudine conſtare cernerem,

evolu-

evolutionem eiusdem pro caſu quodam utili et Tetragonometriae proprio prorſus omiſi, cum nullius vel certe exiguae utilitatis fore in praxi praeviderem, et brevitati ſtuderem. Ex aequationibus interdum theoremata haud inelegantia elicui, et arbitrarias determinationes ſaepius adhibui, per quas problemata ad triangula pertinentia ſolvuntur eleganter, quae aliis methodis non ſolvuntur, quo ipſo methodi Tetragonometricae utilitas in aliis problematibus ſolvendis elucet, verum poſthaec multo magis eluceſcet, ſi quis Pentagonometriae, Hexagonometriae, et in genere Poligonometriae condendae ſe applicuerit.

In hunc tractatum tetragonometricum tres terminos in Geometria prius non receptos introduxi; neceſſarium enim fuit, ut me ſatis

apte

apte et concinne exprimerem, a vulgari et ordinario trapezio, quod utramque diagonalem habet intra ſe, alterum illud diſtinguere, minus uſitatum, quod alteram vel utramque diagonalem habet extra ſe. His itaque figuris ex natura ſua nomen impoſui, et trapezium inverſum appellavi; partialiter inverſum, quod alteram, totaliter inverſum, quod utramque diagonalem habet extra ſe, et in relatione et oppoſitione ad hoc trapezium, vulgare illud et uſitatum directum vocavi; haec autem diſtinctio aptis terminis expreſſa neceſſaria videbatur.



INDEX CAPITUM.

INTRO-

INTRODUCTIO

IN

TETRAGONOMETRIAM.



INTRODUCTIO
IN
TETRAGONOMETRIAM.

CAPUT I.

Recenſio et examen fundamentorum Tetragonometriae V. Clariſſ. Lambert.

§. 1.

Univerſalis poſſibilitas, quamlibet figuram quadrilateram rectilineam in duo triangula rectilinea per diagonalem dividendi, ei cogitationi haud dubie occaſionem dedit, quod qui in calculo Trigonometrico bene ſe exercuerit; is etiam in figurarum quadrilaterarum calculo ſine difficultate progredi poſſet. Cui opinioni quid et quantum ſubſit veri, ex hac recenſione patebit, in qua contentus ero, omnes caſus poſſibiles Tetragonometriae recenſere et problematum eo ſpectantium numerum definire.

§. 2. Si quadrilaterum rectilineum per ſe conſideretur, ſine diviſione per diagonalem in triangula, praeter latera quatuor ac totidem angulos nihil amplius habet. Quare tum haec occurrit quaeſtio quomodo aliquae harum partium

 per

per caeteras determinentur, et quot casus sint possibiles? Et cum in Algebra, quando ad aequationem est perveniendum, inter quantitates datas et quaesitas non consideretur discrimen, ad abbreviandum etiam ego hoc discrimen seponam, quippe quod magis aequationum resolutionem, quam inventionem concernit.

§. 3. Cum in omni quadrilatero rectilineo, summa omnium angulorum sit quatuor rectis, sive 360° aequalis, ex tribus angulis datis quartus per se inveniri potest. Data vero a se invicem independentia sint oportet, idcirco ex quatuor angulis plusquam ex tribus non reperitur; consequenter in aequatione quaesita plures quam ad summum tres anguli contineri non debent.

§. 4. Sed tres anguli cum duobus lateribus totum triangulum determinant, ideoque duorum laterum reliquorum uno in aequationem insuper adscito, obtinetur haec propositio: Tres anguli et tria latera se se mutuo determinant; quam tamen sic intelligendam quilibet videt, qui Geometriam elementarem, vel a primo limine salutavit, quod trium laterum ac totidem angulorum, quinque simul sumta, sextum determinent. Cum igitur in hoc casu angulus unus et latus unum in aequatione desint, respectu positionis angulorum duo casus occurrunt; potest enim angulus absens adjacere lateri absenti vel non, ac ideo inter duo latera quae in aequatione adsunt, intercipi. Cum autem respectu aequationis nihil intersit, inter quaenam bina latera ex *illis tribus dictus* angulus interjaceat; superfluum omnino foret hunc casum ulterius subdividere,

videre, consequenter pro tribus lateribus et tribus angulis, non habentur nisi duo casus.

§. 5. Si unus angulus ex aequatione dimittatur, hic defectus uno latere supplendus est, et sic in aequatione habentur duo anguli et quatuor latera, horum singulis quinis sextum determinantibus, atque etiam hic respectu positionis angulorum duo dantur casus speciales, qui facile cogitari possunt: duo enim isti anguli vel diagonaliter sibi opponuntur, vel eidem lateri adjacent. Habentur ergo quatuor casus universales et quatuor aequationes eos exprimentes, quarum quaelibet sex continet casus particulares, pro numero determinationum in ipsa contentarum, et cum eorum quilibet ut quaesitum spectari possit, ideo pro quatuor aequationibus habentur casus viginti quatuor.

§. 6. Hi viginti quatuor casus in quatuor primis figuris (Fig. 1. 2. 3. 4.) repraesentantur, prima namque figura continet quatuor latera et duos angulos diagonaliter oppositos; et aequatio eo pertinens Cl. Lamberti est hujusmodi:

$$a^2 + b^2 - c^2 - d^2 = 2ab \cos \omega - 2cd \cos \psi.$$

Secunda figura continet quatuor latera et duos angulos eidem lateri adjacentes, et aequatio harum sex partium relationem exprimens, quam dedit Cl. Lambert est talis

$$a^2 + b^2 + c^2 - d^2 = 2a\ (b \cos \omega + \cos \phi) - 2bc \cos (\omega + \phi)$$

Tertia figura continet tria latera et tres angulos, duos eidem lateri adjacentes et unum binis e tribus reliquis interceptum, quarum sex partium relationem exprimit haec aequatio Cl. Lambert:

$$a \,Sin.\, \omega = c \,Sin.\, (\omega + \phi) + d \,Sin.\, \lambda.$$

 Quarta

Quarta continet ſex partes quae ſe invicem continuo conſequuntur, quarum relationem exprimit haec aequatio a Cl. Lambert data:
$c\,Sin.\psi = a\,Sin.(\varphi+\psi) - b\,Sin.(\varphi+\psi+\omega)$.

§. 7. In quatuor iſtis aequationibus omnes anguli aſſumuntur acuti. Quod ſi deinceps occurrant caſus in quibus ſint obtuſi, ſive ſummae angulorum, $\omega+\varphi$, $\varphi+\psi$, $\varphi+\psi+\omega$, majores duobus rectis, tum ſigna rite erunt immutanda, quod tamen tum demum erit faciendum, cum aequatio fuerit reſoluta, quia in plerisque reſolutionibus functiones angulorum immutantur. Caeterum apparet non omnes ſolutiones aeque faciles et ſimplices eſſe, et in ſpecie, quod omnes ſex ſolutiones aequationis ſecundae praebeant valores duplices irrationales.

§. 8. Omnes hi caſus viginti quatuor, qui in quatuor iſtis aequationibus continentur, poſſunt in ipſa praxi geodaetica occurrere, quando campi, agri vel ſtagni cujusdam ichnographia ex peripheria eſt conficienda, dum neque omnes anguli, neque omnia latera propter obſtacula varia aeque facile menſurae ſubjici poſſunt. Sunt autem hi caſus viginti quatuor, nequaquam omnes; ſed illi ſoli qui occurrere poſſunt, quando res peragenda ſolum per angulos et latera. Quod ſi vero triangulum per diagonalem in duo triangula diſpeſcatur, multis modis data et quaeſita ſic disponi poſſunt, ut neutrum triangulum, ſed ut ambo triangula ſimul ſumenda ſint, ut ex iis, quae in utroque ſimul data ſumuntur, ad quaeſitum pateat aditus, quae conditio efficit, ut illi caſus proprie loquendo non ad Trigono-

metri-

metriam, ſed ad Tetragonometriam pertineant, atque ideo ſibi proprias ſolutiones et difficultates habeant.

§. 9. Hos caſus a figura quinta usque ad quadrageſimam ſecundam repreſentavit Cl. Lambert, ut poſt laborioſam recenſionem ſe invenisse et in ordinem redegiſſe dicit. Quaelibet harum figurarum habet aequationem ſibi propriam, et cum in quavis ſex res occurrant, quarum quaelibet ut quaeſitum ſpectari poteſt: oriuntur hinc pro figuris triginta octo, ſexies triginta octo, hoc eſt 228 diverſi caſus, de quibus ſequentes fecit annotationes auctor. Nimirum in qualibet harum figurarum notantur ſex partes, quae aequationem ingredi debent, et pleraeque lineola ſignantur, quando autem occurrit totus angulus, per quem diagonalis tranſit, arculus adhibetur. In qualibet figura unus ad minimum horum angulorum integer occurrit, quo ipſo duo triangula a ſe invicem dependentia efficiuntur. Ex eadem ratione praeter diagonalem adeſt alter angulorum eidem adjacentium, aut etiam occurrit uterque; in neutro triangulo plura quam tria prorſus determinata eſſe debent, quoniam alias in uno triangulo determinantia nimis multa, in altero nimis pauca haberentur.

§. 10. Figuras eo ordine deſignauit auctor, quo ſunt recenſendae, quo ipſo earundem deſcriptio decurtatur. Angulus diagonalis inferior in omnibus occurrit, et praeterea quinque notantur partes; et cum multae permutationes ſint poſſibiles, ad eas in claſſes digerendas a diagonali eſt orſus, qui occurrit in figuris 5ta, uſque

A 3 ad

ad 25^{tam}, in ſequentibus autem non occurrit. Ambae claſſes principales totidem caſus continent, ſi primi quatuor caſus claſſi poſteriori annumerentur.

§. 11. Primam claſſem principalem dividit auctor in quatuor minores, quarum prima continet figuras 5^{tam}, 6^{tam}, 7^{mam}. Secunda continet figuras 8^{tavam}, 9^{nam}, 10^{mam} et habet eundem angulum in duos partiales per diagonalem diviſum, qui diverſa nomina ſortiuntur et ambo aequationem ingrediuntur. Tertia claſſis continet figuras, 11^{mam} uſque ad 19^{nam}, quae non habet niſi alterum horum angulorum. Quarta continet figuras 20^{mam} uſque ad 25^{tam} et habet neutrum; et quidem figura 20ma quatuor latera, figura 21ma, 22da tria latera et unum angulum. Figura 23^{tia}, 24ta, 25ta, duo latera et duos angulos.

§. 12. Secunda claſſis principalis, in qua diagonalis abeſt, iterum intuitu anguli ſuperioris ad diagonalem in duas claſſes ſubdividitur: unam a figura 26ta uſque ad 31^{am}, alteram a 32^{a} uſque ad 42^{am}; vel ſi ſpecialiter ſit procedendum in ſex claſſes ſubdividitur, quarum

I. continet	Fig.	26,	27,	28.
II. - -	Fig.	29,	30,	31.
III. - -	Fig.	32,	33,	34.
IV. - -	Fig.	35,	36,	37.
V. - -	Fig.	38,	39,	40.
VI. - -	Fig.	41,	42.	

Et hactenus fundamenta Tetragonometriae Cl. Lamberti, ipſius fere verbis recenſuimus niſi ubi aliquid majoris claritatis ergo paululum immutavimus vel adjecimus. Supereſt ut hanc ipſius recenſionem accuratiori examini ſubjiciamus.

§. 13.

§. 13. Ratiocinium Cl. Lambert, quo universam Tetragonometriam in duas classes principales, et priorem iterum in quatuor; posteriorem in duas vel etiam sex minores dispescuit, ita fere se habet. Quoniam angulus ad diagonalem inferior in omni casu adesse debet ut problema solvendum, non ad Trigonometriam, sed Tetragonometriam pertineat, in casu solvendo aut insuper adest diagonalis, aut non adest. Si prius obtinetur classis prior: nimirum Systema omnium problematum Tetragonometricorum, in quibus diagonalis in numero sex determinationum aequationem ingredientium continetur; Si posterius, habetur classis posterior, nimirum Systema omnium problematum, in quibus diagonalis non deprehenditur in numero sex illarum determinationum ex quibus aequatio conficitur.

§. 14. Sumo prius classem priorem examinandam, et in omni problemate hujus classis adsunt duae determinationes, nimirum; angulus combinationis et diagonalis; adest igitur praeterea angulus ad diagonalem superior vel prorsus abest, quae disjunctiva propositio classem priorem in duas subdividit, quarum prioris character est habere angulum ad diagonalem superiorem, posterioris, eodem prorsus carere. Igitur in omni problemate classis prioris habentur haec tria: diagonalis et angulus ad ipsam uterque, ita tamen ut superior nondum sit satis determinatus, cum adesse possit indivisus vel divisus. Si prius; obtinetur prima classis earum quatuor, in quas auctor priorem principalem dispescuit, quare in omni problemate hujus classis adsunt tres determinationes, scilicet angulus combinationis, dia-

gonalis et angulus ad ipſum ſuperior indiviſus, quae aequationem ingredi debent, reliquis tribus nondum determinatis. Haec autem tria ad utrumque triangulum eodem modo referuntur, nimirum angulus uterque ad neutrum triangulum pertinet, diagonalis ad utrumque. Verum tres determinationes reliquae inter duo triangula ſic erunt tribuendae, ut facta determinatione in uno triangulo non adſint niſi tria determinata, diagonali connumerata; et in altero non pauciora quam duo, diagonali etiam ad ipſum relato. Sic enim ſex habentur determinationes, quae aequationem ingrediuntur.

§. 15. Sumatur jam triangulum alterutrum ex. gr. ſiniſtrum (arbitrarium enim omnino eſt utrumvis triangulum ad hoc eligatur) et in ipſo fiant duae determinationes. Tria in utroque triangulo adſunt, nimirum duo latera et angulus interceptus, quae primo certe intuitu adhiberi poſſe videntur, et conſequenter tres determinandi modi poſſe uſurpari combinando ſingula bina, ſcilicet vel duo latera, vel latus ſuperius et angulum, vel etiam latus ſiniſtrum et angulum. Sed re attentius perpenſa non habentur niſi duo determinandi modi, qui poſſint pro diverſis haberi et ad aequationes ſpecie diuerſas perducant, nimirum latus utrumque et latus alterutrum cum angulo intercepto; quippe quemadmodum latus ſuperius et angulus adjacens reſpiciunt angulum combinationis inferiorem; ita latus ſiniſtrum et idem angulus reſpiciunt angulum totalem ſuperiorem; atque ideo hi caſus rectius confunduntur in unum, cum ad aequationes diverſas non perducant.

§. 16.

§. 16. Ex duobus itaque determinandi modis in triangulo finiſtro, aſſumatur prior, puta latera duo, et additio ſextae determinationis e triangulo dextro fieri nequit niſi duobus modis, ſumendo ſcilicet vel angulum vel latus alterutrum; Sumtio enim lateris dextri et infimi, non ducit ad aequationes formaliter diverſas, cum utrumque latus ad angulos diagonaliter oppoſitos ſimiliter referatur, quare ſumtio lateris dextri et infimi rectius confunduntur in unum. Sumtio itaque amborum laterum in triangulo ſiniſtro, non ſuppeditat niſi duos caſus, quos exhibent figurae quinta et ſeptima. Jam in eodem triangulo aſſumatur modus determinandi alter, ſcilicet latus alterutrum cum angulo, atque ſtatim patet, omittendum eſſe angulum in triangulo dextro, niſi enim omitteretur quatuor in aequatione forent anguli, quod fieri non debere author inde ab initio monſtravit. Ergo ſexta determinatio in dextro triangulo ſumenda eſt, latus alterutrum, neque enim ob rationes modo allatas ſumtio lateris dextri et infimi ad aequationes ſpecie diverſas perducit. Ex hac itaque ſumtione oritur caſus tertius, quem figura ſexta repraeſentat. In hac itaque prima claſſe, in qua diagonalis et anguli diagonaliter oppoſiti adſunt, tanquam character ſpecieficus, tres tantum continentur caſus, qui ad aequationes formaliter et ſpecie diverſas perducunt.

§. 17. Habuimus claſſem in qua angulus ad diagonalem ſuperior adeſt indiviſus, examinemus jam eam, in qua adeſt diviſus, quae in duas denuo ſubdividitur, in quarum prima adeſt uterque partialis, in altera tantum alteruter. Sumamus

mus primo priorem, et apparet ſtatim quatuor haberi determinationes, nimirum angulum combinationis, diagonalem et utrumque partialem ſuperiorem ad diagonalem. Itaque duae deſunt, quas ita ſumendas eſſe patet, ut una in dextrum altera in ſiniſtrum triangulum cadat. Primo certe intuitu duo latera quatuor modis ſumi poſſe videntur, nimirum. 1) Latera conjuncta circa angulum combinationis. 2) Conjuncta circa angulos partiales. 3) Disjuncta ſupremum et infimum, 4) Disjuncta dextrum et ſiniſtrum. Sed re accuratius perpenſa ultimi duo caſus non ducunt ad aequationes formaliter et ſpecie diverſas, propterea quod bina latera hoc modo ſumta, angulum combinationis et partiales eodem modo reſpiciunt, itaque duo iſti caſus confunduntur in unum hoc modo enunciandum, ut dicatur, eſſe is, qui obtinetur ſumendo latera disjuncta. Adeoque tres tantum obtinentur caſus, qui claſſem ſecundam ſub priori generali contentam ſecundum Cl. Lambert abſolvunt et figûris 8va 9na et 10ma repraeſentantur. Sed hic auctor unum caſum omiſiſſe videtur, qui prodit ſumendo latus utrumvis in triangulo uno, et angulum diagonali oppoſitum in triangulo altero, quem caſum an recte rejiciat poſtea diſpiciam.

§. 18. Abſoluto examine ſecundae claſſis ſub priore generali contentae, pervenimus ad tertiam, cujus hic eſt character, quod alter tantum adſit angulus ad diagonalem ſuperior, arbitrarium vero omnino eſt in utrovis triangulo ſumatur. Capiatur ergo in dextro uti fecit Cl. Lambert, jam tres habentur determinationes, deficientibus aliis tribus, quarum duae ſumendae ſunt in triangulo ſini-

ſiniſtro, cum duae habeantur, quae poſſunt ad dextrum referri, ſcilicet diagonalis et angulus partialis, ſed non niſi una, quae poſſit ad ſiniſtrum referri, puta eadem diagonalis. Primo quidem intuitu quatuor ſumtiones adhiberi poſſe videntur, 1) trium laterum, 2) duorum laterum et anguli unius, 3) duorum angulorum et lateris unius, 4) denique trium angulorum, puta duorum diagonaliter oppoſitorum et tertii partialis ad diagonalem ſuperioris. Sed quia hic ex hac claſſe excluditur, ſumtio quarta locum non habet. Sumantur ergo primo tria latera, et quidem duo in triangulo ſiniſtro, unum in dextro, ergo ſumi poteſt vel dextrum, quem caſum exhibet fig. 12^{ima}, vel infimum, quem repraeſentat fig. 15ta Sunt autem hi caſus revera diverſi, quia poſitio lateris ad angulum combinationis eſt diverſa, atque ideo ad aequationes formaliter et ſpecie diverſas deducunt. Sumatur ſecundo in triangulo ſiniſtro latus ſuperius cum angulo et cum hac ſumtione conjungatur latus dextrum vel infimum in triangulo dextro, quos caſus repraeſentant figurae 11. et 14. Sumatur tertio latus infimum cum angulo, cum qua ſumtione combinetur, ut prius, latus alterum in dextro triangulo, quos caſus exhibent fig. 13ta et 16ta. Singulis ſumtionibus in triangulo ſiniſtro jungenda reſtat ſumtio anguli, qui diagonali opponitur in dextro, quos tres caſus exhibent figurae 17ma, 18va, 19na, atque ita in univerſum novem caſus in claſſe tertia continentur, ſub principali priori comprehenſa.

§. 19. Hae fuere tres claſſes ſub priore principali contentae, in quibus angulus ad diagona-

gonalem ſuperior adeſt totaliter vel partialiter; ſupereſt claſſis quarta, in qua prorſus abeſt, habentur ergo duae determinationes, nimirum, angulus combinationis et diagonalis, deficientibus quatuor, quarum binae in utroque triangulo ſumendae. Scilicet vel quatuor laterum, quem caſum exhibet fig. 20ma vel trium laterum et anguli unius; et ſic in triangulo ſiniſtro ſumtis duobus, ſumi poteſt vel dextrum vel infimum in dextro triangulo, una cum angulo adjacente, quos duos caſus exhibent fig. 21ma, 22da. Sumtio duorum laterum in triangulo dextro et lateris tertii cum angulo in ſiniſtro, non perducunt ad caſus a prioribus formaliter et ſpecie diverſos, quandoquidem latus cum angulo adjacente in utroque triangulo eodem modo refertur ad angulum combinationis. Poſſunt denique ſumi duo latera et duo anguli, qui diagonali opponuntur, et quidem vel latera conjuncta circa angulum combinationis, vel conjuncta circa angulum ſuperiorem ad diagonalem, vel denique duo latera disjuncta, quae ſumtio primo intuitu in duas ſubdividi poſſe videtur, cum disjunctiva ſint latera vel ſupremum et infimum vel etiam dextrum et ſiniſtrum. Sed haec ſubdiviſio non perducit ad aequationes formaliter et ſpecie diverſas, quare hi caſus rectius confunduntur in unum, unde tres obtinentur caſus, qui repraeſentantur figuris 23ta, 24ta et 25ta. Igitur ſex caſus continet claſſis quarta ſub priore principali contenta, cujus nunc examen ad umbilicum perductum.

§. 20. Claſſis principalis poſterior ſubdividitur, ſumendo determinationes ſpecificas ab angulo

gulo ſuperiore; etenim ex numero ſex illarum determinationum quae ſemper adeſſe debent, exulante diagonali, vel uterque vel alter tantum adeſt angulus partialis, non enim amplius in hac claſſe ponere licet, eundem adeſſe indiviſum, cum hac ratione rediret res ad duos angulos diagonaliter oppoſitos et quatuor latera, qui caſus erat omnium primus figura prima repraeſentatus. Neque jam ſumere licet eundem angulum abeſſe totaliter, ſic enim recurret ille caſus, in quo habentur duo anguli adjacentes et omnia latera, qui caſus figura ſecunda fuit exhibitus. Sumatur ergo jam primum, uterque angulus partialis adeſſe, ut una cum angulo combinationis tres determinationes habeantur. Ad tres determinationes, quae deſiderantur ſupplendas, ſumi poſſunt vel latera tria, vel latera duo vel angulus unus. Videtur addi poſſe ſumtio duorum angulorum et lateris unius, ſed hanc ſumtionem adhibere non licet, ſic enim omnes anguli in quadrilatero ſimul adhiberentur, quod fieri non debere auctor inde ab initio monſtravit. Sumantur ergo primo tria latera, in dextro quidem triangulo duo, in ſiniſtro unum; ita tamen ut arbitrarium omnino ſit, in utrovis binarius ſumatur; cum haec diverſitas ad aliquam aequationum differentiam formalem non deducat; ſumtis ergo duobus lateribus in dextro, ſumi poteſt vel ſupremum vel ſiniſtrum in ſiniſtro, quos duos caſus exhibent figurae 27ma et 30ma. Adhibeantur ſecundo latera duo, & angulus unus, ſumendo in dextro triangulo latus unum cum angulo adjacente, et quidem vel dextrum vel infimum. Fiat ſumtio prior, et in ſiniſtro ad-

adhibeatur ſumtio lateris alterutrius, quos duos caſus exhibent figura 28va et 31ma; fiat ſumtio poſterior, et ut prius addi poteſt ex ſiniſtro triangulo latus alterutrum, quos caſus exhibent figurae 26ta et 29na. Videtur quidem eadem ſumtio lateris cum angulo adhiberi poſſe in triangulo ſiniſtro, addendo ſumtionem lateris alterutrius in dextro; verum hae ſumtiones non perducunt ad aequationes formaliter et ſpecie diverſas, quia latera cum angulo adjacente eodem modo reſpiciunt angulum combinationis et partialem ad diagonalem ſuperiorem. In hac itaque claſſe, in qua ponitur, uterque partialis ad diagonalem ſuperior adeſſe, ſex continentur caſus, nec plures eſſe poſſunt.

§. 21. Supereſt claſſis poſterior, ſub altera principali tanquam partialis contenta, cujus hic eſt character, quod alter tantum adſit angulus partialis ad diagonalem ſuperior, qui ſumi poteſt vel in dextro vel in ſiniſtro triangulo, ſed arbitrarium eſt in utrovis ſumatur. Sumatur ergo in dextro, quo ſumto, quatuor deſunt determinationes, quarum binas in utroque ſumendas eſſe patet; poſſunt ergo ſumi quatuor latera, quem caſum exhibet fig. 39na vel tria latera et unus angulus. Sumantur ergo duo latera in triangulo ſiniſtro, cui ſumtioni jungatur in dextro ſumtio lateris alterutrius cum angulo, quos caſus exhibent figurae 33^{a} et 36^{a}; poſſunt etiam ſumi duo latera et anguli duo, qui diagonali opponuntur; ex quo ſequitur unum in triangulo ſiniſtro eſſe ſumendum et cum ipſo latus alterum conjungendum. Quod ſi ergo ſumatur latus alterutrum cum angulo in ſiniſtro, et jungatur huic ſumtioni

latus

latus alterutrum cum angulo in dextro, quatuor oriuntur casus, quos exhibent figurae 32da, 34ta, 35ta, 37^{ima}. Sumta sunt prius tria latera et angulus unus, sed ita, ut latera duo sumta fuerint in sinistro triangulo. Ad latus reliquum cum angulo in dextro potest vero eadem sumtio repeti sine confusione ac identitate casuum, et id quidem ita, ut jam duo latera sumantur in dextro, latus vero cum angulo in sinistro, et quidem vel dextrum cum angulo, vel etiam supremum cum angulo, quos casus exhibent figurae 38va et 40ma. Denique tria latera et angulus unus ita sumi possunt ut tres determinationes, hoc est latera duo cum angulo intercepto cadant in triangulum sinistrum, reliqua determinatio scilicet latus in dextrum, et quidem vel in infimum vel in dex trum, quos duae figurae ultimae 41ma et 42da repraesentant. Caeterum hic videtur Cl. Lambert casum unum omisisse, qui oritur, si, quemadmodum prius sumantur in sinistro duo latera cum angulo intercepto, atque huic sumtioni jungatur sumtio anguli in dextro; num autem hic casus recte sit praetermissus infra dispiciam.

§. 22. Classem principalem posteriorem Clariss. Lambert etiam aliter disposuit, divisione facta in sex classes minores, et ut prius ponendo utrumque angulum partialem adesse. Pro tribus determinationibus, quae desiderantur, sumit duas in triangulo dextro, in sinistro unam, et quidem primo latus superius, quo sumto, duae determinationes tribus modis sumi possunt in dextro, vel enim accipitur latus suppremum cum angulo, vel latus utrumque, vel denique latus dextrum cum angulo, quos casus repraesentant.

fig.

fig. 26ta, 27ma, 28va, atque his tribus casibus absolvitur classis prima. Secundam classem determinat sumtione lateris sinistri in triangulo sinistro, cum qua sumtione conjunguntur ut prius tres sumtiones in dextro, hi tres casus, quibus secunda classis absolvitur, exhibentur figuris 29na, 30ma, 31ma. Atque haec erat subdivisio classis prioris duarum sub una principali contentarum. Classis posterior, in qua non adest nisi alter partialis in dextro triangulo, harum primum determinat sumendo latus cum angulo in dextro triangulo, cumqua sumtione conjungit latus alterutrum cum angulo, vel etiam utrumque in sinistro, quos casus exhibent figurae 32da, 33ta, 34ta quibus absolvitur tertia classis minor, prima vero earum quatuor, quae sub ea classe continentur, in qua non habetur nisi alter angulus ad diagonalem partialis. Quartam classem determinat sumendo in dextro latus supremum cum angulo, et cum hac sumtione combinando easdem ac prius determinationes, quos casus repraesentant figurae 35ta, 36ta 37ma, atque haec est secunda classis quatuor posteriorum. Quintam classem auctor determinat sumendo in triangulo dextro duo latera, et cum hac sumtione eadem ac ante in sinistro combinando, quos casus exhibent figurae 38va, 39na, 40ma quibus absolvitur tertia classis, sub posteriore contentarum, alterum partialem habente. Sextam denique classem determinat, sumendis in triangulo sinistro tribus determinationibus, et cum hac sumtione combinando latus alterutrum in triangulo dextro, quos duos casus exhibent duae ultimae figurae, atque haec est quarta ultimaque classis earum quatuor, quae continentur sub

ſub claſſe poſteriore unum angulum ad diagonalem partialem habente.

§. 23. Ita ſe ergo habent fundamenta Tetragonometriae, quae poſuit Clariſſ. Lambert, atque inſtituto examine, nihil habeo validum et momentoſum, quod ipſi opponam, niſi forte ſint tres illi caſus ſupra (§. 17. 21.) memorati, ab eo forte non ſatis recte praetermiſſi. Illud profecto mihi certum, hos caſus deducere ad aequationes formaliter et ſpecie diverſas; cur autem eas omiſerit auctor, cauſam videre non poſſum, niſi forte putarit, aequationes iſtas fore in Tetragonometria prorſus inutiles, id eſt inter ſex illos caſus, qui in aequatione tetragonometrica continentur, nullum eſſe Tetragonometriae proprium; ſed ſolvi trigonometrice expeditius, quod an ita revera ſe habeat, in progreſſu hujus tractatus diſpiciam, cum numerus et ordo problematum me eo deduxerit.

Sunt autem problemata tetragonometrica duorum generum, unum genus eſt eorum problematum, quae methodo tetragonometricae ita ſunt propria, ut trigonometrice proprie loquendo minime ſolvantur; alterum genus eſt eorum, quae Tetragonometriae propria non ſunt; ſed utrique methodo et tetragonometricae et trigonometricae ſubſunt, et id quidem ita, ut plerumque brevius et expeditius trigonometrice quam tetragonometrice ſolvantur.

Prioris generis problemata recte vocantur tetragonometrica primaria; poſterioris generis ſecundaria. Caeterum omnia problemata tetragonometrica ſub hoc uno generaliſſimo

comprehenduntur: in figura quadrilatera rectilinea, ex datis quinque construere figuram.

§. 24. Caeterum dedit auctor tantum quatuor aequationes, pro quatuor primis casibus, in quibus figura ex perimetro et angulis est construenda, quas nudas proposuit, non ostensa methodo ad ipsas perveniendi, neque quicquam ex iis concludendo. Igitur ab ovo ordiendum putavi, et has quatuor aequationes, tanquam abessent denuo inveniendas; ut modus ad eas perveniendi appareret. Sed ut supra dictum, in numerum illarum determinationum quae aequationes ingrediuntur, nihil admittitur praeter quatuor latera et angulos, in quorum numero sunt duo partiales ad diagonalem, nisi ipsa diagonalis. Possunt enim longe plures determinationes adhiberi, quibus adhibitis, numerus problematum in magnam multitudinem excresceret. Verum quemadmodum in Trigonometriam stricte sic dictam, nullae determinationes admittuntur praeter angulos et latera; ita etiam in Tetragonometria non adhibentur determinationes nisi modo dictae. Atque sicuti in theoria triangulorum, adhibitis aliis determinationibus quam angulis et lateribus, scientia illa potius Trigonoscopia quam Trigonometria nuncupatur: ita etiam Tetragonometria, pluribus aliis determinationibus adhibendis rectius Tetragonoscopia dicitur.

CAPUT

CAPUT II.

De quatuor primis casibus Tetragonometriae V. C. Lambert, in quibus figura ex perimetro et angulis est construenda.

Definitiones.

§. 25. Tetragonometria est scientia ex quinque in quadrilatero rectilineo datis, sextam determinationem inveniendi. Angulus combinationis vel connexionis est angulus ad diagonalem inferior finistimus, qui semper ut indivisus consideratus utrumque triangulum connectit.

Scholion.

§. 26. Ex Geometria elementari lectori notum supponitur, figuram quadrilateram rectilineam ex datis quinque determinationibus, angulis, lateribus et diagonali sufficienter determinari et construi posse. Sed arbitrarium omnino est angulum ad diagonalem inferiorem, pro angulo combinationis eligere; verum commoditas positionis respectu figurae et oculi intuentis suadet, ut prae caeteris eligatur. In toto vero hoc tractatu, angulos arculis, latera et diagonalem, commate designo. In toto hoc capite denominationes Clariss. Lambert retineo, et in principio angulos, qui aequationem ingredi debent, cum illo semper acutos adhibeo.

Problema I.

Fig. I. §. 27. In figura quadrilatera $ABCD$, inter quatuor latera: $AA=a$, $BC=b$, $CD=d$, $AD=c$, et angulos diagonaliter oppoſitos $B=\omega$, et $D=\psi$, aequationem invenire

Solutio.

1) Inter angulos reliquos A et C, ducatur diagonalis AC, a cujus extremis in latera AB, et CD, demittantur normales Cc et Aa.

2) Cum utrumque triangulum ſit acutangulum per hypotheſin et conſtructionem; patet, per Euclidis Elem. II. 2. eſſe $AB^2+BC^2-2AB.BC=AC^2=AD^2+CD^2-2CD.aD$.

3) Cum ſit $BD=b\ cos.\ \omega$ et $aD=c\ cos.\ \psi$; erit, ſymbolis ſubſtitutis, $a^2+b^2-2ab\ cos.\ \omega = c^2+d^2-2cd\ cos.\ \psi$, ipſa aequatio Lambertiana, et facta transpoſitione $a^2+b^2-c^2-d^2=2ab\ cos.\ \omega-2cd\ cos.\ \psi$ ut ſupra §. 6. *Q.E.J.*

Coroll. I.

§. 28. Cum omnia latera aequationi eodem modo inſint, etiam omnia per aequationes quadraticas affectas ſimiliter exprimuntur, hoc modo,

$AB=a=\pm b\ cos.\omega+\sqrt{(c^2+d^2\mp 2cd\ cos.\psi-b^2 fin.\omega^2)}$

$BC=b=\pm a\ cos.\omega+\sqrt{(c^2+d^2\mp 2cd\ cos.\psi-a^2 fin.\omega^2)}$

$CD=d=\pm c\ cos.\psi+\sqrt{(a^2+b^2\mp 2ab\ cos.\omega-c^2 fin.\psi^2)}$

$AD=c=\pm d\ cos.\psi+\sqrt{(a^2+b^2\mp 2ab\ cos.\omega-d^2 fin.\psi^2)}$

Scholion I.

§. 29. Si angulus extra ſignum radicale ſit obtuſus, ſigno negativo utendum; at ſi obtuſus exiſtat

existat angulus sub signo radicali, hoc est si triangulum obtusangulum sit, signum affirmativum adhibendum. Signa radicalia negativa ut inutilia rejiciuntur.

Coroll. II.

§. 30. Pro angulis determinandis est

$$\cos.\ \omega = \frac{a^2+b^2-c^2-d^2 \pm 2cd\cos.\psi}{2ab} \text{ nec non}$$

$$\cos.\ \psi = \frac{c^2+b^2-a^2-b^2 \pm 2ab\ \cos.\omega}{2cd} \text{ et hinc}$$

$$\omega = \text{ang. } \cos.\frac{(a^2+b^2-c^2-d^2+2cd\cos.\psi)}{2ab} = B$$

$$\psi = \text{ang. } \cos.\frac{(c^2+d^2-a^2-b^2 \pm 2ab\ \cos.\omega)}{2cd} = D.$$

Coroll. III.

§. 31. Quod si in figura quadrilatera proposita duo anguli diagonaliter oppositi sint aequales duobus rectis, et consequenter figura circulo inscriptibilis, erit

$$\cos.\ \omega = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)} \text{ et } \cos.\psi = \frac{c^2+d^2-a^2-b^2}{2(ab+cd)}$$

ut posterior sit prior negative sumtus, existente affirmativa expressione anguli acuti; obtusi negativa, sit $a^2+b^2=A^2$, $c^2+d^2=B$. ut sit

$$\cos.\ \omega = \frac{A^2-B^2}{2(ab+cd)} = \frac{(A+B)(A-B)}{2(ab+cd)} \text{ quae expres-}$$

sio sequens suppeditat.

 Theo-

Theorema.

In quadrilatero circulo inſcriptibili, dupla ſumma rectangulorum ex lateribus angulos diagonaliter oppoſitos continentibus, eſt ad rectangulum ex ſumma et differentia diagonalium eorundem, ut ſinus totus ad coſinum anguli intercepti.

Scholion II.

§. 32. Inter ſex illos caſus, qui in hac aequatione tetragonometrica continentur, nullus datur Tetragonometriae proprius, qui non facilius et ſimplicius trigonometrice ſolvatur, ita ut in caſibus omnibus Trigonometria Tetragonometriae palmam praeripiat. Nam ſi Ichnographia figurae ſit conficienda et tres anguli propter impedimenta quaedam obſervari nequeant, ex uno angulo obſervato et lateribus omnibus datis, ſemper trigononometrice figura conſtrui poteſt, et id quidem facilius et ſimplicius quam tetragonometrice. Idem obtinet, ſi ex duobus angulis obſervatis et tribus lateribus datis figura ſit conſtruenda, itaque conſtructiones hujus problematis tetragonometricas omitto, et ad ſequens problema feſtino.

Problema II.

Fig. II. 1. §. 33. In figura quadrilatera $ABCD$ inter quatuor latera $AB=a$, $BC=b$, $CD=d$, et $AD=c$, et angulos duos adjacentes lateri AB, $B=\omega$, et $A=\varphi$, aequationem invenire

Solutio.

Solutio.

1) Ab angulis C et D, in latus AB, demittantur normales Cc et Da, et ab angulo D demittatur normalis Dd in perpendiculum Cc, et erunt per principia geometrica Cc et Da inter se parallelae et recta Dd aequalis rectae ca.

2) Quoniam est $BC = b \cos.\omega$, et $Aa = c \cos.\varphi$; erit $ac = a - b \cos.\omega - c \cos.\varphi = Dd$ et quia est $Cc = b \sin.\omega$, atque $Da = c \sin.\varphi$, erit $Cd = b \sin.\omega - c \sin.\varphi$.

3) Quia angulus aDd rectus est, et consequenter triangulum CdD rectangulum; erit per theorema Pythagorae $CD^2 = Cd^2 + Dd^2$, ut substitutis symbolis habeatur,
$d^2 = (a - b \cos.\omega - c \cos.\varphi)^2 + (b \sin.\omega - c \sin.\varphi)^2 = a^2 - 2ab \cos.\omega + b^2 \cos.\omega^2 - 2ac \cos.\varphi - 2bc \sin.\omega \sin.\varphi + 2bc \cos.\omega \cos.\varphi + c^2 \sin.\omega^2 + c^2 \cos.\omega + b^2 \sin.\omega$.
sed cum per principia analytica trigonometrica sit $\sin.\varphi^2 + \cos.\varphi^2 = \sin.\omega^2 + \cos.\omega^2$, et $\cos.\omega \cos.\varphi - \sin.\omega \sin.\varphi = \cos.(\omega + \varphi)$; factis substitutionibus aequatio in hanc contrahitur: $d^2 = a^2 + b^2 + c^2 - 2ab \cos.\omega - 2ac \cos.\varphi + 2bc \cos.(\omega + \varphi)$ adeoque habetur ut supra (§. 6.) $a^2 + b^2 + c^2 - d^2 = 2a(b \cos.\omega + c \cos.\varphi) - 2bc \cos.(\omega + \varphi)$ ipsissima aequatio Clariss. Lambert.

Coroll. I.

§. 34. Ex hac aequatione obtinentur singula latera per aequationes quadraticas affectas hoc modo

latus $AB == a == \pm b \cos. \omega \pm c \cos. \varphi \pm$
$\pm \surd (d^2 - (c \sin. \varphi - b \sin. \omega)^2)$
$BC == b == \pm a \cos. \omega \pm c \cos. (\omega + \varphi)$
$\pm \surd (d - (a \sin. \omega - c \sin. (\omega + \varphi))^2)$
$AD == c == \pm a \cos. \varphi \mp b \cos. (\omega + \varphi)$
$\pm \surd (d^2 -- (a \sin. \varphi -- b \sin. (\omega + \varphi))^2)$
$CD == d == \surd (a^2 + b^2 + c^2 \pm 2b (c \cos. (\omega + \varphi)$
$\mp a \cos. \omega) \mp 2ac \cos. \varphi)$

Scholion I.

§. 35. Inter ſex illos caſus ſive quaeſtiones, quae in aequatione tetragonometrica continentur, non datur niſi una Tetragonometriae propria, quae tunc occurrit, quando ex tribus datis lateribus et angulis duobus quaeritur latus illud cui adjacent, et hic conſtructio figurae tetragonometrice fit; caeteri omnes parum habent momenti, et expeditius trigonometrice conſtruuntur. Sed etiamſi per Trigonometriam hic caſus non conſtruatur, tamen per Geometriam elementarem independenter a Tetragonometria tam ſimpliciter conſtruitur, ut cum conſtructione Tetragonometriae de ſimplicitate certet.

Conſtructio tetragonometrica eſt hujus modi:

Fig. II. 2. 1) Ducatur recta MN aequalis lateri dato CD, et ſuper ea conſtruatur ſemicirculus.

2) Transferantur anguli dati B et A in M et N, et producantur crura datorum angulorum donec productae rectae MP et NQ ſint aequales datis lateribus BC et AD, et ab extremis in rectam MN demiſſis perpendiculis PT et QV, quorum hoc eſt $= c \sin. \varphi$, illud $= b \sin. \omega$, capiatur

tur eorum differentia $PR = c\,\mathit{fin}.\,\varphi - b\,\mathit{fin}.\,\omega$, et ab extremo diametri applicetur chorda MO ipſi aequalis, et ducatur chorda complementi $NO =$ $= \surd(d^2 - (c\,\mathit{fin}.\,\varphi - b\,\mathit{fin}.\,\omega)^2)$.

3) Jam in linea indefinita AB capiatur ac $= NO$, et in punctis a et c excitentur normales cC, aD, antea inventis perpendiculis PT et QV aequales, a quorum extremis C et D ad lineam indefinitam AB applicentur latera data BC et AD, et connectantur puncta, C et D, et conſtructa erit figura tetragonometrice. Eſt enim vi conſtructionis $cB = b\,cos.\,\omega$, $aA = c\,cos.\,\varphi$ et $ca = \surd(d^2 - (c\,\mathit{fin}.\,\varphi - b\,\mathit{fin}.\,\omega)^2)$ et conſequenter erit latus quaeſitum $AB =$ $= a = b\,cos.\,\omega + c\,cos.\,\varphi + \surd(d^2 - (c\,\mathit{fin}.\,\varphi - b\,\mathit{fin}.\,\omega)^2)$ Ex conſtructione patet ſignum radicale negativum eſſe rejiciendum, ſed pro coſinibus ſigna affirmativa valent, ſi anguli ſint acuti; quod ſi vero ſint obtuſi, negativa valebunt, quo in caſu poſitio contraria illius cB, et aA in linea ac ſumitur, hoc eſt, ab eadem ſubtrahuntur; et uno angulo ex. gr. B, acuto, altero A obtuſo exiſtente cB lineae ac additur, aA vero ab eadem ſubtrahitur. Fig. II. 3.

Scholion II.

§. 36. Haec erat conſtructio, quam Tetragonometria ſuppeditat; ſed altera quam Geometria elementaris ſine Tetragonometriae ſubſidio ſuppeditat talis eſt.

1) In linea indefinita Ba fac angulum $CBa =$ angulo dato B, ejusdemque anguli crus BC aequale lateri dato BC. Eodem modo Fig. II. 4.

 con-

conſtituatur angulus datus A verſus finem lineae Ba, ita ut ſit daB aequalis angulo dato A, et fiat crus ad aequale dato lateri AD.

2) Poſito circino in C extendatur ad diſtantiam dati lateris CD, et ducatur arcus EF indefinite intra latera data BC et ad, vel etiam ſi placet integer ſemicirculus.

3) Per punctum d uſque ad arcum EF ducatur recta GH lineae Ba parallela, et a puncto interſectionis D ad C ducatur recta DC; quo facto per punctum interſectionis D agatur AD parallela lineae ad et conſtructa erit figura quaeſita $ABCD$. Cum enim Dd per conſtructionem ſit parallela Aa vel aB, et AD rectae ad, erit angulus BAD aequalis angulo Bad, et AD aequalis ad. Sed per conſtructionem eſt CD aequalis lateri dato tertio, conſequenter figura conſtructa ea eſt, quae quaerebatur. Caeterorum caſuum, cum trigonometrice facilius fere expediantur, conſtructiones omitto. Si quis autem ſcire velit quomodo anguli ex aequatione eruantur et exprimantur, et ex. gr. quaeratur angulus $B=\omega$, tum ad abbreviandum poſito $\text{ſin.}\,\varphi=m$, $\cos.\,\varphi=n$, et $a^2+b^2+c^2-d^2-2acn = M^2$, $\text{ſin.}\,\omega = x$ et $\cos.\,\omega = \sqrt{(1-x^2)}$ factis ſubſtitutionibus obtinetur haec aequatio

$$x = \frac{M^2cm \pm (a-cn)\sqrt{(4b^2(a^2+c^2-2acn)-M^4)}}{2b\,(a^2+c^2-2acn)} = \text{ſin.}\,\omega$$

et cum uterque angulus aequationi ſimiliter inſit, anguli etiam φ ſinus ſimili aequatione exprimitur.

Pro-

Problema III.

§. 37. In propofita figura quadrilatera $ABCD$, inter haec fex, latus $AB=a$ $AD=c$, $CD=d$, angulum $A=\varphi$, $B=\omega$, et $C=\lambda$; aequationem invenire Fig. III. 1.

Solutio.

1) Ab angulo A in latus BC, demittatur normalis Ab, quae igitur erit $= a\ fin.\ \omega$, et ab angulo D in idem latus perpendicularis $Dc = d\ fin.\ \lambda$.

2) Ab angulo D demittatur etiam normalis in perpendiculum Ab, et erit Ad differentia perpendiculorum Ab et Dc per principia geometrica, et confequenter $= a\ fin.\ \omega - d\ fin.\ \lambda$.

3) Per punctum A agatur lateri BC parallela, et producatur ad ipfam cD, et erit per principia Geometriae elementaris $cg = Ab$, et praeterea angulus $DAg = \pi - \omega - \varphi$, et confequenter $fin.\ DAg = fin.\ (\omega+\varphi)$ et hinc $Dg = Ad = c\ fin.\ (\omega+\varphi) = a\ fin.\ \omega - d\ fin.\ \lambda$, adeoque habetur haec aequatio Lambertina. $a\ fin.\ \omega = c\ fin\ (\omega+\varphi) + d\ fin.\ \lambda$. Q. E. I.

Coroll. I.

§. 38. Ex hac aequatione fimpliciffima valores laterum ftatim obtinentur, cum fit

$$\text{latus } AB = a = \frac{c\ fin.(\omega+\varphi) + d\ fin.\ \lambda.}{fin.\ \omega}$$

$$AD = c = \frac{a\ fin.\ \omega - d\ fin.\ \lambda.}{fin.(\omega+\varphi)}$$

$$DC = d = \frac{a\ fin.\ \omega - c\ fin\ (\omega+\varphi)}{fin.\ \lambda}$$

Coroll.

Coroll. II.

§. 39. Sed anguli hoc modo exprimuntur, cum ſit $fin. (\omega + \phi) = \frac{a\, fin.\, \omega - d\, fin.\, \lambda}{c}$, erit ſumma angulorum $\omega + \phi = Arc. \frac{(fin. \omega - d\, fin. \lambda)}{c}$ et conſequenter $\phi = Arc. \frac{fin. (fin.\, \omega - d\, fin.\, d)}{c} - \omega$.

Sed erit $fin.\, \lambda = \frac{a\, fin.\, \omega - c\, fin.\, (\omega + \phi)}{d}$ conſequenter $\lambda = Arc. \frac{fin. (a\, fin.\, \omega - c\, fin.\, (\omega + \phi))}{d}$

Coroll. III.

§. 40. Angulus vero ω, cum aequationi aliter inſit, hoc modo exprimi nequit. Sit igitur ad abbreviandum $fin.\, \phi = m$, $cos.\, \phi = n$, $fin.\, \lambda = l$, $fin.\, \omega = x$, $cos.\, \omega = \surd(1 - x^2)$ et factis ſubſtitutionibus habetur haec aequatio: $(a - c\, n)\, x - d\, l = c\, m \surd(1 - x^2)$ ex qua operatione abſoluta obtinetur haec aequatio,

$$x = \frac{d\, l\, (a - cn) \pm m\, c \surd(a^2 + c^2 - 2\, a\, c\, n - d^2 l^2)}{a^2 + c^2 - 2\, a\, c\, n}$$

Scholion I.

§. 41. Inter ſex illa problemata particularia, quae in aequatione continentur, non dantur niſi duo Tetragonometriae propria, quorum unum obtinet, quando ex reliquis quinque quaeritur angulus A, alterum ſi quaeratur latus eidem adjacens AD, et ex utrovis conſtructio figurae.

Caete-

Caetera quatuor utrique methodo et Tetragonometriae et Trigonometriae fubfunt; ita tamen ut trigonometrice fere facilius et naturalius folvantur; horum igitur omiffa conftructione, duorum Tetragonometriae propriorum adfero conftructionem, quae hoc modo adornatur.

1) Crus BA anguli B prolongetur donec fit aequale lateri dato $AB=a$, et ab extremo A demittatur in crus alterum Bm indefinite productum normalis $Ab=a\,fin.\,\omega$. Fig. III. 2.

2) In crure Bm ad diftantiam indefinitam Bm conftituatur angulus nmB aequalis angulo dato C et crus mn producatur donec fit aequale lateri dato CD.

3) Ab hujus extremo n in perpendiculum Ab, demiffa normali nd, fi angulus A quaeratur, detur vero latus AD, applicetur hoc in A ad perpendiculum modo demiffum, et in A conftructus erit angulus quaefitus; Si vero angulus A detur, conftituatur is in AB verfus dextram, et producatur crus AD ufque ad perpendiculum nd, et in utroque cafu per punctum concurfus D, acta parallela cum recta mn, conftructa erit figura. Nam ab D demiffum perpendiculum in $BC=d\,fin.\,\lambda$, et $Ad=a\,fin.\,\omega-d\,fin.\,\lambda$, fed, acta per A parallela lateri BC, et ad hanc producto perpendiculo cD, erit pars producta $=Dg=Ad=c\,fin.(\omega+\varphi)=a\,fin.\,\omega-d\,fin.\,\lambda$.

Problema IV.

§. 42. In propofita figura quadrilatera $ABCD$, inter tria latera $AB=a$, $BC=b$, $AD=c$, et tres angulos $A=\varphi$, $B=\omega$, et $D=\psi$; aequationem invenire Fig. IV. 1.

Solutio.

Solutio.

1) Ex angulis A et B in latus DC productum in c demitte normales Ad et Bc, per punctum A age lateri DC parallelam donec cum perpendiculo cB producto concurrat, et per principia Geometriae elementaris, erit Bb, normalis ad Ab, et recta $bc = Ad$, ex his sequitur angulum BAb esse $= \pi - \varphi - \psi$, cujus ergo sinus est $\mathit{fin.}(\varphi + \psi)$ et hinc $Bb = a\,\mathit{fin.}(\varphi + \psi)$

2) Cum sit angulus $C = 2\pi - \varphi - \omega - \psi$; erit $\mathit{fin.}\,C = -\mathit{fin.}(\varphi + \omega + \psi)$, consequenter erit $Bc = -b\,\mathit{fin.}(\omega + \varphi + \psi)$. Sed cum sit $Ad = c\,\mathit{fin.}\,\psi$; pervenitur ad hanc aequationem, $c\,\mathit{fin.}\,\psi = a\,\mathit{fin.}(\varphi + \psi) - b\,\mathit{fin.}(\omega + \varphi + \psi)$ quae est ipsa aequatio Clariss. Lambert.

Coroll. I.

§. 43. Ex hac aequatione omnia latera expedite obtinentur, cum sit

$$a = \frac{c\,\mathit{fin.}\,\psi + b\,\mathit{fin.}(\omega + \varphi + \psi)}{\mathit{fin.}(\varphi + \psi)} = AB$$

$$\text{et } b = \frac{a\,\mathit{fin.}(\varphi + \psi) - c\,\mathit{fin.}\,\psi}{\mathit{fin.}(\omega + \varphi + \psi)} = BC$$

$$\text{nec non } c = \frac{a\,\mathit{fin.}(\varphi + \psi) - b\,\mathit{fin.}(\omega + \varphi + \psi)}{\mathit{fin.}\,\psi} = AD$$

Coroll. II.

§. 44. Angulus B per hanc aequationem obtinetur $\omega = \mathit{ang.}\frac{(a\,\mathit{fin.}(\varphi + \psi) - c\,\mathit{fin.}\,\psi)}{b} - \varphi - \psi$.

Sed

Sed angulus D per tangentem obtinetur, ut ſit $tang.\,\psi = \frac{a\,fin.\,\varphi - b\,fin.\,(\omega+\varphi)}{c - a\,cos.\,\varphi + b\,cos.\,(\omega+\varphi)}$ et conſequenter ipſe angulus $\psi = ang.\,tang.\left(\frac{a\,fin.\,\varphi - b\,fin.\,(\omega+\varphi)}{c - a\,cos.\,\varphi + b\,cos.\,(\omega+\varphi)}\right)$

Coroll. III.

§. 45. Poſito ad abbreviandum $fin.\,\omega = m$, $cos.\,\omega = n$, $fin.\,\psi = h$. $fin.\,(\varphi+\psi) = x$, $cos.\,(\varphi+\psi) = \sqrt{(1 - x^2)}$, dato enim $fin.\,(\varphi+\psi)$, cum datus ponatur $fin.\,\psi$, datur etiam $fin.\,\varphi$, factis ſubſtitutionibus obtinetur haec aequatio abbreviata: $b\,m\,\sqrt{(1 - x^2)} = (a - bn)x - ch$, ex qua operatione abſoluta, obtinetur

$$x = \frac{ch(a-bn) \pm bm\sqrt{(a^2+b^2-2abn-c^2h^2)}}{a^2+b^2-2abn} = fin.(\varphi+\psi)$$

et conſequenter ipſe angulus

$$\varphi = ang.\,fin.\,\frac{((a-bn)\,ch \pm b\,m\sqrt{(a^2+b^2-2abn-c^2h^2)})}{a^2+b^2-2abn} - \psi$$

Coroll. IV.

§. 46. Quod ſi latera AB et CD ponantur eſſe parallela, ut prodeat trapezium parallelarum baſium, aequatio primo inventa mutatur in hanc: $c\,fin.\,\psi = b\,fin.\,\omega$, ex qua ſequitur haec analogia, $c : b = fin\,\omega : fin.\,\psi$, quae ſequens ſuppeditat.

Theorema.

In trapezio parallelarum baſium, latera parallelis intercepta ſunt reciproce ut ſinus angulorum diagonaliter oppoſitorum; cujus theorematis veritas ſynthetice ſtatim perſpicitur.

Scho-

Scholion I.

§. 47. Inter ſex problemata particularia, quae in aequatione tetragonometrica continentur, non eſt niſi unum Tetragonometriae proprium, quod tunc obtinet, cum ut incognitum quaeritur latus AB, et conſtructio figurae; caeteri quinque caſus fere facilius et expeditius trigonometrice ſolvuntur. Caſus autem Tetragonometriae proprii haec eſt conſtructio.

Fig. IV. 2. 1) Conſtituatur in D angulus datus ψ, et prolongetur crus unum DA ad longitudinem dati lateris AD, alterum vero indefinite verſus g et in puncto g conſtituatur angulus obtuſus trium datorum complementum ad quatuor rectos, cujus crus gf producatur ad longitudinem dati lateris BC.

2) Per hujus extremum f agatur lateri DC parallela, uſque ad latus AD, et in A conſtituatur angulus datus φ, cujus crus AB prolongetur ad concurſum cum ducta modo parallela.

3) Niſi punctum concurſus B inciderit in extremitatem rectae fg, per idem agatur parallela ad rectam fg, et conſtructa erit figura. Eſt enim $Ad = c$ *ſin.* ψ, et $Bc = -b$ *ſin.* $(\omega+\varphi+\psi)$, et acta lateri DC parallela per A, uſque ad productum perpendiculum bc, erit $Bb = a$ *ſin.* $(\varphi+\psi) = Aa$ conſequenter c *ſin.* $\psi = a$ *ſin.* $(\varphi+\psi) - b$ *ſin.* $(\omega+\varphi+\psi)$.

Scholion II.

§. 48. Pertractatis hucusque quatuor primis Tetragonometriae problematibus animadvertere coepi, aequationes ad quas pervenitur contemplando

plando ſaltim trapezium directum, quod utramque diagonalem includit, ſine omni immutatione haud univerſales eſſe, et pro trapezio ſive partialiter ſive totaliter inverſo non valere, niſi in angulis vel terminis vel etiam utrisque aliqua ſignorum commutatio contingat; ideoque conſtitui hic, ſicuti in ſequentibus problematibus, in adjecto ſcholio commonſtrare, quomodo aequationes immutandae ſint, ut pro trapezio non directo modo, verum etiam inverſo quocunque valeant et generaliſſimae fiant. Primam igitur aequationem Cl. Lambert quod attinet, illa quidem pro generali haberi poteſt, quae pro omni trapezio tam directo quam inverſo valeat, cum ex ipſa natura coſinus ſequatur, eum tam affirmativum quam negativum eſſe poſſe, tamen opus non eſt ut ſignum utrumque ſemper coſinui praeponatur; ſufficit enim id ſignum in aequatione exprimi, quod ex ſpecie figurae aſſumtae et aequationis reductione ſequitur, atque hac ratione aequationis primae Clariſſ. Lambert univerſalitas facile adſtruitur, quae utroque ſigno expreſſo ita ſe haberet:
$a^2 + b^2 - c^2 - d^2 = \pm 2\,ab\cos.\omega \mp 2\,cd\cos.\psi.$

Scholion III.

§. 49. Aequatio Cl. Lambert ſecundi problematis valet tantum pro trapezio directo, ſiquidem anguli A et B ſint ambo acuti, ſi vero in trapezio directo ambo hi anguli ſint obtuſi valet haec aequatio: $a^2 + b^2 + c^2 - d^2 = -2a\,(c\cos.\varphi + b\cos.\omega) - 2\,bc\cos(\omega + \varphi)$. Quod ſi angulus A ſit obtuſus, B vero acutus,

C habe-

habetur haec aequatio: $a^2 + b^2 + c^2 - d^2 =$
$= 2a\,(b\,cos.\,\omega - c\,cos.\,\phi) + 2bc\,cos.\,(\phi - \omega)$
Denique ſi angulus A ſit acutus, B vero obtuſus; valet haec aequatio: $a^2 + b^2 + c^2 - d^2 =$
$= 2a\,(c\,cos.\,\phi - b\,cos.\,\omega) + 2bc\,cos.\,(\omega - \phi)$
Generalis igitur aequatio pro omni trapezio directo eſt hujus modi: $a^2 + b^2 + c^2 - d^2 = \pm$
$\pm\, 2a\,(c\,cos.\,\phi \pm b\,cos.\,\omega) \mp 2bc\,cos.\begin{pmatrix}\omega \pm \phi\\ \phi - \omega\end{pmatrix}$

Si trapezium ſit partialiter inverſum, ut latus AD cadat intra latus AB, et ſimul latus CD intra latus CB, iisdem angulis acutis exiſtentibus, valet ſine omni immutatione aequatio Lambertina. Sed angulo B obtuſo, A acuto exiſtente, valet aequatio tertia ſine mutatione. Si trapezium ſit totaliter inverſum, cadente latere AD intra latus AB, CD vero extra CB, et angulo B acuto exiſtente, valet aequatio Lambertina; Verum exiſtente utroque angulo acuto, et latere AB intra latus AD, CD vero intra latus CB valet haec aequatio: $a^2 + b^2 + c^2 - d^2 =$
$= 2a\,(c\,cos.\,\phi + b\,cos.\,\omega) - 2bc\,cos.\,(\omega + \phi)$, in hoc eodem caſu exiſtente angulo B obtuſo, valet eadem aequatio cum ſigno affirmativo termini dextimi in dextro membro. Quod ſi angulus A ponatur obtuſus, et B acutus, ſive trapezium ſit partialiter ſive totaliter inverſum; valet aequatio ſecundo allata pro trapezio directo. Generaliſſima igitur aequatio pro omni trapezio tam directo quam inverſo, eſt illa ipſa quam ſupra attuli pro trapezio directo.

Verum aequationis Lambertinae quoad terminum priorem dextri membri univerſalitas uti ſupra ſine ſignis defendi poteſt, cum ex natura coſi-

cosinuum sequatur, quatuor signa duobus cosinibus apponi posse; Similiter ex natura cosinus sequitur posteriori termino etiam utrumque signum praeponi posse, tamen quoad angulos in hoc eodem termino aequationis Lambertinae universalitas defendi non potest.

Scholion IV.

§. 50. Aequatio tertia Lambertina valet etiam invariata pro trapezio partialiter inverso, cadente latere AD intra latus AB, et latere CD intra CB. Si vero latus AD cadat extra latus AB, et latus CD extra latus CB, prodit haec aequatio: $a \, sin. \, \omega = c \, sin. \, (\omega - \phi) - d \, sin. \, \lambda$, si vero latus AD cadat intra latus AB, sed latus CD extra latus CB, valet haec aequatio: $a \, sin. \, \omega = c \, sin. \, (\omega + \phi) - d \, sin. \, \lambda$. Quod si denique latus AD cadat extra latus AB, latus CD intra latus CB, habetur haec aequatio: $a \, sin. \, \omega = c \, sin. \, (\omega - \phi) + d \, sin. \, \lambda$. Aequatio igitur generalis haec erit: $a \, sin. \, \omega = c \, sin. \, (\omega \pm \phi) \pm d \, sin. \, \lambda$.

Scholion V.

§. 51. In quarto problemate existente trapezio directo et obtinente angulo sive C sive D obtuso, caeteris, quorum dantur nomina primitiva, acutis, valet aequatio Lambertina. Si adjacentes lateri $B\,C$ anguli B et C sint ambo obtusi; et si adjacentes lateri AB anguli A et B, sint ambo obtusi, valet invariata aequatio Clariss. Lambert. Si anguli A et C diagonaliter oppositi sint obtusi, et si angulus B sit obtusus, caeteri acuti, valet invariata aequatio Lambertina.

 Deni-

Denique si angulus A sit acutus, caeteri omnes obtusi; eadem aequatio invariata locum habet. His igitur omnibus casibus pro trapezio directo valet haec aequatio Clariss. Lambert: $c\ fin.\ \psi = a\ fin.\ (\phi+\psi) - b\ fin.\ (\phi+\psi+\omega)$. Verum in trapezio directo, praeter hos casus dantur aliquot, in quibus sine omni immutatione non valet, nam si anguli diagonaliter oppositi B et D sint obtusi, valet haec aequatio: $c\ fin.\ \psi = - a\ fin.\ (\phi+\psi) - b\ fin.\ (\omega+\phi+\psi)$; eadem aequatio valet, si anguli adjacentes lateri AD, A et D sint ambo obtusi; existente angulo A obtuso, caeteris omnibus acutis, eadem aequatio locum habet, Existente angulo D obtuso et caeteris acutis, et existente angulo B obtuso, caeteris acutis, et existente angulo C acuto, caeteris obtusis, denique existente angulo D acuto, caeteris obtusis; valet haec aequatio eadem, quare generalis aequatio pro trapezio directo haec erit: $c\ fin.\ \psi = \pm a\ fin.\ (\phi+\psi \mp)\ b\ fin.\ (\phi+\psi+\omega)$. Si trapezium sit partialiter inversum, existente angulo B acuto, et latere AD intra latus AB, latere CD intra latus CB cadente, et angulo D obtuso, obtinet haec aequatio: $c\ fin.\ \psi = a\ fin.\ (\psi-\phi) - b\ fin.\ (\psi-\omega-\phi)$. Sin autem latus CD cadat extra latus CB, ut trapezium sit totaliter inversum, sive angulus D sit obtusus vel acutus, prodit haec aequatio: $c\ fin.\ \psi = a\ fin.\ (\psi-\phi) + b\ fin.\ (\omega+\phi-\psi)$; Si, caeteris manentibus iisdem, latus AD cadat extra latus AB, et latus CD intra latus BC, existente trapezio totaliter inverso, prodit haec aequatio: $c\ fin.\ \psi = a\ fin.\ (\phi+\psi) - b\ fin.\ (\phi+\psi-\omega)$. Si angulus D sit acutus et B obtusus, cadat vero totum

totum triangulum ADC extra triangulum ABC, ut trapezium ſit partialiter inverſum, habetur haec aequatio: $c\ \mathit{ſin.}\ \psi = a\ \mathit{ſin.}\ (\phi + \psi) + b\ \mathit{ſin.}\ (\omega - \phi - \psi)$. Si angulus B ſit acutus, eadem aequatio prodit. Si angulus D ſit acutus, B vero obtuſus, cadente latere AD extra latus AB, et latere CD, intra latus CB, exiſtente trapezio totaliter inverſo, prodit haec aequatio pro trapezio inverſo tertia. Si manentibus caeteris iisdem fiat angulus B acutus, eadem aequatio prodit; quod ſi vero latus AB cadat extra latus AD, latus CB intra latus CD, ut trapezium ſit totaliter inverſum, exiſtente uti prius angulo D acuto, ſed angulo B ſive acuto ſive obtuſo, valet aequatio pro trapezio inverſo ſecunda. Si angulus B ſit obtuſus et angulus D etiam obtuſus, ſitque trapezium totaliter inverſum, ut latus AB cadat extra latus AD, et latus CB intra latus CD; obtinet etiamnum eadem aequatio, et quicunque caſus praeterea cogitari poſſunt, in iis valet aliqua aequationum hactenus allatarum. Igitur generaliſſima aequatio pro omni trapezio tam directo quam inverſo erit hujusmodi: $c\ \mathit{ſin.}\ \psi = \pm\ a\ \mathit{ſin.}\ (\psi \pm \phi)$ $b\ \mathit{ſin.}\ (\pm \psi \mp \omega \mp \phi)$

Scholion VI.

§. 52. Hactenus fuerunt quatuor illa problemata, quorum aequationes dedit ipſe Cl. Lambert. Jam igitur ad reliqua progrediens, in toto hoc tractatu eandem denominationem laterum adhibeo; angulorum autem nomina ſic immuto, ut angulus combinationis A, mihi ſemper ſit $=\psi$, $B=\lambda$, $C=\omega$, $D=\phi$, angulus partialis $ACB=\alpha$ $ACD=\beta$.

 CAPUT

CAPUT III.

Continens tria problemata primae classis ex quatuor illis, quae sub priore principali continentur.

Problema V.

Fig. V. §. 53. In figura quadrilatera rectilinea proposita $ABCD$ inter haec sex: latus $AB = a$ $BC = b$, $AD = c$, diagonalem $AC = f$, angulum $A = \psi$, et $C = \omega$; aequationem invenire

Solutio.

1) In triangulo ABC ab angulo C in latus AB, demissa normali CD, quaero cosinum anguli B, qui, quoniam acutus est assumtus, invenitur esse $\frac{a^2 + b^2 - f^2}{2ab} = cos. B$, quo ad abbreviandum posito $= h$, erit $sin. B = \sqrt{(1-h^2)}$; in eodem triangulo formo hanc analogiam $AC: sin. B = AB: sin. ACB$, et symbolis substitutis. $f: \sqrt{(1-h^2)} = a: sin. ACB$ consequenter est $sin. AcB = \frac{a\sqrt{1-h^2}),}{f}$ nec non $cos. AcB =$ $= \frac{\sqrt{f^2 - a^2(1-h^2)}).}{f}$

2) Ex his et ex sinu cosinuque anguli C, formo sinum anguli ACD; cum enim sit angulus

gulus $ACD = C - ACB = \omega - ACB$; erit $\sin. ACD = \sin. (\omega - AcB) =$
$$= \frac{\sin. \omega \surd (f^2 - a^2(1-h^2)) - a \cos. \omega \surd (1-h).}{f}$$

3) Formo finum et cofinum fummae trium angulorum A, B, C, et obtineo $\sin. (A + B + C) =$
$= h \sin. (\omega + \psi) + \cos. (\omega + \psi) + \cos. (\omega + \psi) \surd (1-h^2)$
et $\cos. (A + B + C) = h \cos. (\omega + \psi) - \sin. (\omega + \psi)$
$\surd (1 - h^2)$

4) Ex his et ex finu cofinuque quatuor rectorum, formo finum anguli D, in triangulo ADC; cum enim fit $\sin. ((A + B + C + D) -$
$-(A + B + C)) = \sin. (2\pi - (A + B + C)) = \sin. D$;
obtinebitur $\sin. (2\pi - (A + B + C)) =$
$= h \sin. 2\pi \cos. (\omega + \psi) - \sin. 2\pi \sin. (\omega + \psi)$
$\surd (1-h^2) - h \cos. 2\pi \sin. (\omega + \psi) - \cos. 2\pi \cos. (\omega + \psi)$
$\surd (1-h^2) = \sin. D$. Sed propter $\sin. 2\pi = 0$, et $\cos. 2\pi = +1$, fiet $\sin. D = -h \sin. (\omega + \psi) -$
$- \cos. (\omega + \psi) \surd (1-h^2)$.

5) In triangulo ADC jam pervenio ad hanc analogiam, $AC : \sin. D = AD : \sin. ACD$. h. e.
$f : -h \sin. (\omega + \psi - \cos. (\omega + \psi) \surd (1-h^2) =$
$$= c : \frac{\sin. \omega \surd (f^2 - a^2 (1-h^2)) - a \cos. \omega \surd (1-h^2)}{f}$$

adeoque habetur haec aequatio:
$(a \cos. \omega - c \cos. (\omega + \psi)) \surd (1-h^2) = ch \sin. (\omega + \psi) =$
$= + \sin. \omega \surd (f^2 - a^2 (1-h^2))$. Q. E. F.

Solutio altera.

1) Ducta diagonali BD, propter angulum A acutum affumtum, erit illa $= \surd (a^2 + c^2 -$
$- 2ac \cos. \psi)$ et brevitatis gratia $= k$. In triangulo

gulo ADB formo hanc analogiam $BD: \text{fin.}\ A = AD: \text{fin.}\ ABD$. h. e. $k: \text{fin.}\ \psi =$ $= c: \text{fin.}\ ABD = \frac{c\ \text{fin.}\ \psi}{k}$, confequenter $\cos. ABD = \frac{\sqrt{(a^2 + c^2 - 2ac\cos.\psi - c^2\text{fin.}\psi^2)}}{k} =$ $= \frac{a - c\cos.\psi}{k}$

2) In triangulo BCD formo hanc analogiam: $BD: \text{fin.}\ C = BC: \text{fin.}\ BDC$, hoc eft $k: \text{fin.}\ \omega = b: \text{fin.}\ BDC = \frac{b\ \text{fin.}\ \omega}{k}$ et hinc

$\cos. BDC = \frac{\sqrt{(k^2 - b^2\text{fin.}\omega^2)}}{k}$

3) Hinc formo finum et cofinum fummae angulorum C et BDC, et obtinetur $\text{fin.}(C + BDC) =$ $= \frac{\text{fin.}\omega\sqrt{(k^2 - b^2\ \text{fin.}\ \omega^2)} + b\ \text{fin.}\ \omega}{k} = \text{fin.}\ DBC.$

et $\cos.(C + BDC) = \frac{-\cos.\omega\sqrt{(k^2 - b^2\text{fin.}\omega^2)} + b^2\text{fin.}\omega^2}{k} =$ $= \cos. DBC$

4) Ex datis jam nominibus derivativis angulorum partialium ABD et DBC, formo cofinum anguli totalis B, qui igitur eft

$$\frac{a\cos.\omega\sqrt{(k^2 - b^2\ \text{fin.}\ \omega^2)} + ab\ \text{fin.}\ \omega^2 +}{k^2}$$

$$\frac{+ c\cos.\omega\ \cos.\psi\sqrt{(k^2 - b^2\ \text{fin.}\ \omega^2)}}{k^2}$$

$$\frac{bc\ \text{fin.}\ \omega^2\cos.\psi - c\ \text{fin.}\omega\ \text{fin.}\psi\sqrt{(k^2 - b^2\ \text{fin.}\omega^2)} -}{k^2}$$

$\frac{-bc\ \text{fin.}\omega\cos.\omega\ \text{fin.}\psi}{k^2}$, confequenter obtinetur haec aequa-

aequatio: $h\,k^2 == (c \cos. (\omega + \psi) -- a \cos. \omega) \surd (k^2 -- b^2 \sin. \omega^2) + b \sin. \omega\ (a \sin. \omega --$ $-- c \sin. (\omega + \psi))$ Q. E. I.

Coroll. I.

§. 54. Sit ad abbreviandum in aequatione priore $c \sin. (\omega+\psi) = n$, $a \cos. \omega - c \cos. (\omega+\psi) = m$, et $\sin. \omega = p$, et factis substitutionibus, erit aequatio abbreviata hujusmodi $= 2\,m\,n\,h \surd (1-h^2) ==$ $== (m^2 - n^2 + a^2 p^2)\,h^2 + p^2\,(f^2 - a^2) - m^2$, et omni operatione absoluta ad hanc aequationem pervenio exprimentem cosinum.

$$\cos. B = \frac{\surd (m^2 (m^2 + n^2 + a^2 p^2) -- (m^2 - n^2 + a^2 p^2)}{\surd (m^2+n^2+a^2p^2)^2 - (4a^2n^2p^2)}$$
$$\frac{(f^2 - a^2)\,p^2 \pm 2\,m\,n\,p \surd (f^2 m^2 - (f^2 p - n^2)(f^2 - a^2)))}{\surd (m^2 + n^2 + a^2 p^2)^2 - (4\,a^2 n^2 p^2)}$$

hinc paulo simplicior pro sinu elicitur haec aequatio

$$\sin. B = \frac{\surd (n^2 (m^2 - a^2 p^2) + (m^2 - n^2 + a^2 p^2)\,f^2 p^2 \mp}{\surd (m^2 + n^2 + a^2 p^2)^2 - (4\,a^2 n^2 p^2)}$$
$$\frac{\mp 2\,m\,n\,p \surd\ (f^2 m^2 - (f^2 p^2 - n^2)(f^2 - a^2))^2)}{\surd (m^2 + n^2 + a^2 p^2)^2 - (4\,a^2 n^2 p^2)}$$

Coroll. II.

§. 55. Cum in toto aequationis dextro membro non contineatur b, haec aequatio exhibet verum valorem pro sinu et cosinu anguli B, independenter a latere adjacente $B\,C = b$, et consequenter pro latere $B\,C$ per compendium quoddam quaeri potest sinus vel cosinus anguli adjacentis B; dato enim alterutro datus etiam latus $B\,C$; ponatur brevitatis gratia totum dextrum

C 5 aequa-

aequationis membrum $=M$ ut sit $h=\frac{a^2+b^2-f}{2ab}=M$, et reducendo obtinetur $b=aM\pm\surd(a^2M^2+f^2-a^2)$. Sed neque alterum latus adjacens $AB=a$, neque diagonalis $AC=f$, ex hac aequatione inveniri potest, quia in dextro aequationis membro continentur.

Coroll. III.

§. 56. Sex omnino dantur casus qui statim perspiciuntur, in quibus signum radicale dextimum evanescit, quorum primus obtinet, si sit $m=a\cos.\omega-c\,\mathit{fin}.(\omega+\psi)=0$, secundus si $n=0=$ $=c\,\mathit{fin}.(\omega+\psi)$, tertius si $\mathit{fin}.\ \omega=p$, quartus si $f^2m^2=(f^2p^2-n^2)(f^2-a^2)$, quintus si $f^2p^2=n^2$, denique sextus quando $f=a$; in duobus postremis, quautitas sub signo radicali fit rationalis, quo ipso signum radicale tollitur; plures tamen fortassis dantur rationalitatis casus, sed quibus determinandis supersedeo. Casu quarto fit

$$\cos.B=\frac{\surd(m^2(m^2+n^2+a^2p^2)--}{\surd((m^2+n^2+a^2p^2)^2-4a^2n^2p^2)}$$
$$\frac{--(m^2--n^2+a^2p^2)(f^2--a^2)p^2)}{\surd((m^2+n^2+a^2p^2)^2-4a^2n^2p^2)}$$

et $$\mathit{fin}.\ B=\frac{\surd(n^2(m^2---a^2p^2)+}{\surd((m^2+n^2+a^2p^2)^2-4a^2n^2p^2)}$$
$$\frac{+(m^2--n^2+a^2p^2)f^2p^2)}{\surd((m^2+n^2+a^2p^2)^2-4a^2n^2p^2)}$$

sed in casu penultimo sit

$$\cos.B=\frac{\surd(m^2(m^2+n^2+a^2p^2)--}{\surd((m^2+n^2+a^2p^2)^2-4a^2n^2p^2)}$$
$$\frac{--(m^2-n^2+a^2p^2)(f^2-a^2)p^2\pm 2fm^2np)}{\surd((m^2+n^2+a^2p^2)^2-4a^2n^2p^2)}$$

et

$$\text{et } \sin. B = \frac{\sqrt{(n^2(m^2 - a^2p^2) +}}{\sqrt{((m^2+n^2+a^2p^2)^2 - 4a^2n^2p^2)}}$$
$$\frac{+ (m^2-n^2+a^2p^2)f^2p^2 \mp 2fm^2np)}{\sqrt{((m^2+n^2+a^2p^2)^2 - 4a^2n^2p^2)}}$$

denique in caſu ultimo ſit.

$$\cos. B = \frac{m\sqrt{(m^2+n^2+a^2p^2 \pm 2fnp)}}{\sqrt{(m^2+n^2+a^2p^2f^2\, 4a^2n^2p^2)}}$$

Coroll. IV.

§. 57. Si ponatur $n = o = c \sin. (\omega + \psi)$ qui erat caſus ſecundus, erunt anguli diagonaliter oppoſiti ſimul ſumti $\omega + \psi = \pi$, h. e. duobus rectis aequales, quo in caſu prodit aequatio pro trapezio circulo inſcriptibili hujusmodi.

$$\cos. B = \frac{\sqrt{(a^2+c^2+2ac\cos.\omega - f^2 \sin.\omega^2)}}{\sqrt{(a^2+c^2+2ac\cos.\omega)}} \text{ vel etiam}$$

$$\cos. B = \frac{\sqrt{(a^2+c^2+2ac\cos.\psi - f^2\sin.\psi^2)}}{\sqrt{(a^2+c^2+2ac\cos.\psi)}} \text{ et}$$

$$\sin. B = \frac{f \sin.\omega}{\sqrt{(a^2+c^2+2ac\cos.\omega}} = \frac{f\sin.\psi}{\sqrt{(a^2+c^2-2ac\cos.\psi}},$$

vel etiam quia $\sqrt{(a^2+c^2 \pm 2ac\cos.\omega)}$ repraeſentat diagonalem BD, coſinus ita exhibetur.

$$\cos. B = \frac{\sqrt{(BD^2 - f^2\sin.\omega^2)}}{BD} = \frac{\sqrt{(BD^2 - f^2\sin.\psi^2)}}{BD}$$

$$\text{et } \sin. B = \frac{f\sin.\omega}{BD} = \frac{f\sin.\psi}{BD} = \frac{AC\sin.\psi}{BD.}$$

hinc ergo oritur hoc Theorema.

In quadrilatero circulo inſcripto diagonales ſunt inter ſe ut ſinus angulorum ipſis oppoſitorum.

Coroll.

Coroll. V.

§. 58. Si sit $m = a\, cos.\, \omega - c\, cos.\, (\omega + \psi) = o$, ut sit $a : c == cos.\, (\omega + \psi) : cos.\, \omega$; fiet

$$cos.\, B == \frac{fin.\, \omega\, \surd\, (f^2 - a^2)}{\surd (a^2 fin.\, \omega^2 - c^2 fin.\, (\omega + \psi^2)} = \frac{fin.\, \omega \surd (f^2 - a^2)}{\surd (a^2 - c^2)}$$

quod si denique ponatur $p = fin.\, \omega = o$, ut sit angulus c duobus rectis aequalis, lateribus BC, CD, in diagonalem BD incidentibus et trapezio in triangulum abeunte; ex eadem aequatione prodibit

$$cos.\, B == \frac{a - c\, cos.\, \psi}{\surd (a^2 + c^2 - 2ac\, cos.\, \psi}$$, uti supra §. 53. fuit inventum; nam in hoc casu angulus B, qui in quadrilatero est totalis abit in partialem ABD consequenter hoc ipso aequatio et operatio tota verificatur.

Coroll. VI.

§. 59. Ex eadem aequatione facile obtinetur

$$\text{latus } AD = c = \frac{a\, cos.\, \omega \surd (1 - h^2) - fin.\, \omega \surd (f^2 - a^2 (1 - h))}{h\, fin.\, (\omega + \psi) + cos.\, (^2 + \psi) \surd (1 - h)}$$

et anguli C *tang.* $\omega ==$

$$= \frac{(a - c\, cos.\, \psi) \surd (1 - h) - ch\, fin.\, \psi}{ch\, cos.\, \psi + \surd (f^2 - a^2 (1 - h) - c\, fin.\, \psi \surd (1 - h^2)}$$

Coroll. VII.

§. 60. Si in aequatione posteriore ponatur coefficiens quantitatis radicalis $\surd (k^2 - b^2 fin.\, \omega^2) = m$, et coefficiens $b\, fin.\, \omega = n$, $fin.\, \omega = p$, et $a^2 - f^2 = q^2$ factis substitutionibus obtinetur haec aequatio abbre-

breviata: $2amb\sqrt{(k^2-b^2p^2)}=k^2q^2+b^2(k^2\pm 2anp)$ ex qua operatione ad finem perducta, obtinetur haec aequatio:

$$b=\frac{k\sqrt{(2a^2m^2 - q^2(k^2\pm 2anp)\pm}}{\sqrt{((k^2\pm 2anp)^2+4a^2m^2p^2)}}$$

$$\frac{\pm 2am\sqrt{(a^2m^2-q^2(k^2\pm 2anp+p^2q^2))})}{\sqrt{((k^2\pm 2anp)^2+4a^2m^2p^2)}}$$

Coroll. VIII.

§. 61. Eadem aequatio posterior ea gaudet praerogativa prae priore, ut ex ea diagonalis satis expedite obtineatur per aequationem quadraticam puram, nam posito ad abbreviandum toto dextro aequationis membro $= N$, habebitur haec aequatio $f=\frac{\sqrt{(k^2(a^2+b^2)-2abN)}}{k}$, id quod ideo fieri potuit, quia f in N non continetur.

Scholion I.

§. 62. Inter sex illos casus qui in aequationibus tetragonometricis continentur, duo sunt Tetragonometriae proprii, qui obtinent, quando quaeritur latus AB vel BC, et ex iis constructio figurae, posterioris casus duas dedi solutiones, unam ex aequatione priore, in qua loco lateris quaesitus ac inventus est cosinus anguli B, alteram ex posteriore, in qua directe quaesitum ac inventum est latus ipsum BC; prioris casus solutio ad praxin utilis ex neutra aequatione potest obtineri, cum aequationes ad octavum gradum vel etiam altius ascendant et ingenti constent multitudine terminorum, cujus ideo casus evolu-

evolutionem consulto omisi, sed casus uterque in Geometria practica suppeditat problema utile novum et simul satis pulchrum. Quatuor casus reliqui utrique methodo et trigonometricae et tetragonometricae subsunt, sed ita ut trigonometrice naturalius et expeditius solvantur. Trium tamen ex his casibus solutiones tetragonometricas dedi, quia ex aequationibus satis breviter et expedite poterunt erui. Quarti casus pro inveniendo angulo $A=\psi$, evolutionem omisi, cum ex aequatione operosius eruatur.

Scholion II.

§. 63. Quod si loco lateris AD, manentibus caeteris iisdem, adhibeatur latus $CD=d$, in triangulo ADC, formatis ut prius sinu et cosinu anguli B, et inde sinu et cosinu anguli BAC invenitur

$$\text{sin.}\, CAD = \frac{\text{sin.}\,\psi \sqrt{(f^2 - b^2(1-h^2))} - b \cos.\,\psi \sqrt{(1-h^2)}}{f}$$

Sed est $\text{sin.}\, D = -- h\, \text{sin.}\, (\omega+\psi) - \cos.\, (\omega+\psi \sqrt{(1-h^2)}$ supra inventus; hinc ad aequationem hanc pervenio: $(b \cos.\,\psi -- d \cos.\,(\omega+\varphi) \sqrt{(1-h^2)} == \\ == dh\, \text{sin.}\,(\omega+\psi) + \text{sin.}\,\psi \sqrt{(f^2 -- b^2(1-h^2))}$. Haec aequatio cum illa superiore in solutione priori allata quoad formam prorsus convenit et ex illa formatur, b pro a, d pro c, ψ pro ω scribendo, in sinu et cosinu anguli C, ut verum esse jam pateat, quod asseruit Cl. Lambert, hos casus analytice consideratos a se invicem non differre, sed ad aequationes formaliter easdem perducere, ergo in literis m, n et p, debitis substitutionibus factis et in dextro aequationis membro, utique loco a scripto b, aequatio superior

rior §. 54. etiam pro hoc casu valebit ut sit:

$$cos.\ B = h = \frac{(\surd m^2(m^2+n^2+b^2p^2) -- (m^2-n^2+b^2p^2)}{\surd((m^2+n^2+b^2p^2) -- 4b^2n^2p^2)}$$

$$\frac{(f^2-b^2)p^2 \pm 2mnp\surd(f^2m^2-(f^2p^2-n^2(f^2-b^2)))}{\surd((m^2+n^2+b^2p^2) -- 4b^2n^2p^2)}$$

Igitur quemadmodum in aequatione superiore in toto dextro aequationis membro nullibi continebatur b, et hinc pro latere BC, inveniendo valuit $cos.\ B$, ita etiam in hac aequatione, cum in toto aequationis dextro membro nullibi contineatur a, cosinus B valet pro inveniendo latere AB, et consequenter per hanc aequationem solvitur ille casus Tetragonometriae proprius, qui per aequationem superiorem commode solvi haud poterat.

Scholion III.

§. 64. Similiter ut methodo posteriore, ad aequationem perveniatur similem posteriori, ex lateribus BC et CD formo valorem diagonalis $BD = \surd(b^2+d^2-2bd\,cos.\,\omega) == k$ brevitatis gratia, ergo erit $fin.\ DBC == \frac{d\ fin.\ \omega}{k}$ et $cos.\ DBC = == \frac{b - d\ cos.\ \omega}{k}$; in triangulo ADB est $fin.\ ADB == \frac{a\ fin.\ \psi}{k}$, et $cos.\ ADB == == \frac{\surd(k^2 -- a^2\ fin.\ \psi^2)}{k}$; hinc erit

$$fin.(A + ADB) == \frac{fin.\ \psi \surd(k^2 -- a^2\ fin.\ \psi^2) +}{k}$$

$$\frac{+\ a\ fin.\ \psi\ cos.\ \psi}{k} === fin.\ ABD$$

et

et $cos. (A + ADB) = cos. ABD =$

$$= \frac{\pm cos. \psi \surd (k^2 - a^2 fin. \psi^2 \mp a fin. \psi^2}{k},$$

hinc in eo caſu quo ſumma angulorum A et ADB, non excedit rectum, habetur haec aequatio, quo in caſu valent ſigna ſuperiora $cos. ABD$, $hk^2 = (b\, cos. \psi - d\, cos. \binom{\psi - \omega}{\omega - \psi})$

$$\surd(k^2 - a^2 fin. \psi^2) \pm a\, fin. \psi (d\, fin. \binom{\psi - \omega}{\omega - \psi} - b\, fin. \psi)$$

Sed in altero caſu, quo ſumma eorundem angulorum recto major, in quo valent ſigna inferiora $cos. ABD$, habetur haec aequatio: $hk^2 = (d\, cos. (\omega + \psi) - b\, cos. \psi) \surd (k^2 - a^2 fin. \psi^2) +$ $+ a\, fin. \psi\, (b\, fin. \psi - d\, fin. (\omega + \psi))$; aequatio igitur generalis ita poteſt exhiberi: $hk^2 = (d\, cos. \binom{\omega \pm \psi}{\psi - \omega} - b\, cos. \psi) \surd (k^2 - a^2 fin. \psi^2) \pm$

$$\pm a\, fin. \psi\, (d\, fin. \binom{\omega \pm \psi}{\psi - \omega} - b\, fin. \psi)$$

cujus forma cum forma ſuperioris, quando ſignum anguli utriusque affirmativum adhibetur §. 54. prorſus convenit, ita ut haec ex illa eliciatur pro a, c, b, ω, ψ, reſpective ſcribendo b, d, a, ψ, ω.

Scholion IV.

§. 65. Si in litteris m, n, p, q, k, fiant ſubſtitutiones legitimae, aequatio pro b, ſupra §. 60. exhibita, valebit hic etiam pro a, ut ſit hic quemadmodum ſupra

$$a = \frac{k \surd (2b^2 m^2 - q^2 (k^2 \pm 2bnp) \pm}{\surd ((k^2 \pm 2bmp)^2 + 4b^2 m^2 p^2)}$$

$$\frac{\pm 2bm \surd (b^2 m^2 - q^2 (k^2 \pm 2bnp + p^2 q^2)))}{\surd ((k^2 \pm 2bmp)^2 + 4b^2 m^2 p^2)}$$

Sequi-

Sequitur insuper ex hac aequatione pari difficultate obtineri b, qua ex superiori obtinetur a; denique sequitur aequationem pro diagonali supra §. 6:. inventam etiam respectu hujus aequationis valere, ut sit factis substitutionibus

$$f = \frac{\sqrt{((a^2+b^2)k^2 - 2abN)}}{k}$$

Scholion V.

§. 66. Quoniam, si in datorum numero contineatur latus infimum AD, nulla datur aequatio utilis ad inveniendum latus $AB=a$, et in altero casu, quo in datorum numero continetur latus dextrum $CD=d$, nulla datur aequatio utilis pro inveniendo latere supremo $Bc=b$, in praxi interdum casus facilior difficiliori subsidio venire potest, quippe si in potestate Geodaetae positum sit, in numerum datorum recipere utrum velit, latus AD vel CD; si inveniendum sit latus AB, eligendum erit latus CD, sed si indagandum sit latus BC, adhibendum erit latus AD. Atque hoc modo remedium adhibetur difficultati lateris per aequationem octavi gradus in praxi inutilem indagandi.

Scholion VI.

§. 67. In hoc problemate aequatio prior valet pro omni trapezio directo, siquidem advertatur terminum $ch\,fin.(\omega+\psi)$. propter h, cosinum anguli B, tam affirmative, quam negative accipi posse. Quod si trapezium sit partialiter inversum, ut totum triangulum ADC, cadat intra triangulum ABC sine omni immu-

tatione eadem aequatio valet. Si vero trapezium ita ſit totaliter inverſum ut latus AD cadat intra latus AB, latus autem CD extra latus CB, obtinetur haec aequatio: $(a \cos. \omega - c \cos. (\psi - \omega)) \surd (1-h^2) = ch \sin. (\psi - \omega) - \sin. \omega) (f^2 - a^2 (1-h^2))$. Quod ſi trapezium ita ſit totaliter inverſum ut latus AD cadat extra latus AB, latus autem CD intra latus CB, prodit haec aequatio: $(a \cos. \omega - - c \cos. (\omega - \psi)) \surd (1-h^2) = ch \sin. (\omega - \psi) - - \sin. \omega \surd (f^2 - a^2 (1-h^2))$. Si trapezium ita ſit partialiter inverſum, ut totum triangulum ABC, cadat intra triangulum ADC, habetur haec aequatio: $(a \cos. \omega - c \cos. (\omega + \psi)) \surd (1-h^2) = - - ch \sin. (\omega + \psi) - \sin. \omega \surd (f^2 - a^2 (1-h^2))$. Haec igitur aequatio ex prima ſtatim ſequitur, ſiquidem ob naturam coſinuum ambo termini dextri membri negative ſumantur. Quare generalis aequatio pro omni trapezio erit talis:

$$\left(a \cos. \omega - c \cos. \left(\begin{matrix}\psi \pm \omega \\ \omega - \psi\end{matrix}\right)\right) \surd (1 - h^2) =$$

$$= ch \sin. \left(\begin{matrix}\psi \pm \omega \\ \omega - \psi\end{matrix}\right) + \sin. \omega \surd (f^2 - a^2 (1-h^2)),$$

ubi ob naturam coſinuum ambiguam duplex ſignum conſulto omitto.

Problema VI.

Fig. VI. §. 68. In figura quadrilatera rectilinea propoſita $ABCD$ inter haec ſex: latus $AB = a$, $CD = d$, diagonalem $AC = f$, angulum $A = \psi$, $B = \lambda$, et $C = \omega$; aequationem invenire.

Solutio.

1) In triangulo ſiniſtro formo hanc analogiam $AC(f) : \sin. B (\sin. \lambda) = AB(a) :$ ſin.

$\text{ſin. } ACB = \frac{a \text{ ſin. } \lambda}{f}$ et conſequenter

$$\cos. ACB = \frac{\sqrt{(f^2 - a^2 \text{ ſin.} \lambda^2)}.}{f}$$

2) Ex datis jam ſinibus et coſinibus angulorum B et ACB, formo ſinum et coſinum anguli tertii CAB et obtineo

$$\text{ſin. } CAB = \frac{\text{ſin.} \lambda \sqrt{(f^2 - a^2 \text{ ſin.} \lambda^2)} + a \text{ ſin.} \lambda \cos. \lambda}{f}$$

et $\cos. CAB = \frac{-\cos. \lambda \sqrt{(f^2 - a^2 \text{ ſin.} \lambda^2)} + a \text{ ſin.} \lambda^2}{f}$

3) Ex his et ex ſinu coſinuque anguli A, formo ſinum anguli $CAD = \text{ſin.} (A - CAB) =$

$$= \text{ſin.} (\psi - CAB = \frac{-\text{ſin.} \psi \cos. \lambda \sqrt{(f^2 - a^2 \text{ ſin.} \lambda^2)} +}{f}$$

$$\frac{+ a \text{ ſin.} \lambda^2 \text{ ſin.} \psi - \cos. \psi \text{ ſin.} \lambda \sqrt{(f^2 - a^2 \text{ ſin.} \lambda)}}{f}$$

$$- \frac{a \text{ ſin. } \lambda \cos. \lambda \cos. \psi}{f}$$

$$= \frac{-\text{ſin.} (\lambda + \psi) \sqrt{(f^2 - a^2 \text{ ſin.} \lambda^2)} - a \text{ ſin.} \lambda \cos. (\lambda + \psi)}{f}$$

4) Ad obtinendum nomen derivativum anguli D, ſubduco data nomina primitiva angulorum A, B, C, a nomine primitivo quatuor rectorum 2π, et obtineo $2\pi - \omega - \lambda - \psi$ et hinc erit $\text{ſin.} D = -\text{ſin.} (\omega + \lambda + \psi)$ his factis in triangulo dextro pervenio ad hanc analogiam $AC : \text{ſin.} D = CD : \text{ſin } CAD$ et ſubſtitutis ſymbolis habetur haec

 f: —

$f : - \sin.(\omega + \lambda + \psi) = d : \frac{-\sin.(\lambda+\psi) \sqrt{(f^2 - a^2 \sin.\lambda^2)} -}{f}$

$\frac{-a \sin.\lambda \cos.(\lambda+\psi)}{f}$ consequenter habetur

$d \sin.(\omega + \lambda + \psi) = \sin.(\lambda + \psi) \sqrt{(f^2 - a^2 \sin.\lambda^2)} +$
$+ a \sin.\lambda \cos.(\lambda + \psi)$ Q. E. I.

Scholion I.

§. 69. Potuit etiam ad eandem plane aequationem perveniri, ex sinu et cosinu angulorum C et ACB, formando sinum et cosinum anguli ACD, et ex his nec non ex sinu cosinuque anguli D formando sinum anguli CAD, qui ut prius hac ratione prodiisset

$= \frac{-\sin.(\lambda+\psi) \sqrt{(f^2 - a^2 \sin.\lambda^2)} -}{f}$

$\frac{-a \sin.\lambda \cos.(\lambda+\psi)}{f}$ et consequenter ad eandem analogiam et aequationem pervenissem.

Coroll. I.

§. 70. Ex aequatione inventa statim sine omni fere operatione sequitur valor lateris DC cum sit

$$d = \frac{\sin.(\lambda+\psi) \sqrt{(f^2 - a^2 \sin.\lambda^2)} + a \sin.\lambda \cos.(\lambda+\psi)}{\sin.(\omega + \lambda + \psi)}$$

et per aequationem quadraticam habetur latus sinistrum AB, cum sit operatione absoluta:

$$a = \frac{d \cos.(\lambda + \psi) \sin.(\omega + \lambda + \psi) \pm}{\sin.\lambda}$$

$$\frac{\pm \sin.(\lambda+\psi) \sqrt{(f^2 - d^2 \sin.(\omega+\lambda+\psi)^2)}}{\sin.\lambda.}$$

Schol.

Scholion II.

§. 71. Ad hanc aequationem etiam perveniſſem, ſi ex formato ſinu et coſinu anguli D, formaſſem primo ſinum et coſinum anguli CAD, ſecundo ex his et ex ſinu coſinuque anguli A, ſinum et coſinum anguli CAB, tertio ex his denique et ex ſinu coſinuque anguli B, ſinum anguli ACB; ſic enim ad hanc analogiam in triangulo ABC perveniſſem $AC : \mathit{ſin.}\, B = AB : \mathit{ſin.}\, ACB$ vel etiam ſi ſecundo formaſſem ſinum et coſinum anguli ACD, et ex his et ſinu coſinuque anguli C, ſinum anguli ACB. Sic enim ad hanc eandem analogiam et aequationem perveniſſem: $a\, \mathit{ſin.}\, \lambda = -\mathit{ſin.}\,(\lambda+\psi) \sqrt{(f^2 - d^2\, \mathit{ſin.}\,(\omega+\lambda+\psi)^2)} + d\, cos.\,(\lambda+\psi)\, \mathit{ſin.}\,(\omega+\lambda+\psi)$, quae igitur eſt altera radix aequationis in praecedenti corollario inventae.

Coroll. II.

§. 72. Angulus ad diagonalem ſuperior invenitur eſſe $\omega = ang.\, \mathit{ſin.} \frac{a\, \mathit{ſin.}\, \lambda\, cos.\,(\lambda+\psi) \pm}{d}$

$$\frac{\pm\, \mathit{ſin.}\,(\lambda+\psi) \sqrt{(f^2 - a^2\, \mathit{ſin.}\, \lambda)} - \lambda - \psi}{d}$$

ſed pro angulo A definiendo habetur

$$tang.\,(\lambda+\psi) = \frac{a\, \mathit{ſin.}\, \lambda - d\, \mathit{ſin.}\, \omega}{d\, cos.\, \omega \mp \sqrt{(f^2 - a^2\, \mathit{ſin.}\, \lambda^2)}}$$

et conſequenter ipſe angulus

$$\psi = ang.\, tang. \frac{a\, \mathit{ſin.}\, \lambda - d\, \mathit{ſin.}\, \omega - \lambda}{d\, cos.\, \omega \mp \sqrt{(f^2 - a^2\, \mathit{ſin.}\, \lambda^2)}}$$

Coroll. III.

§. 73. Sed longitudo diagonalis invenitur esse

$$f = \frac{\sqrt{(a^2 fin.\lambda^2 fin.(\lambda+\psi)^2 + (d\,fin.(\omega+\lambda+\psi) -}}{fin.(\lambda+\psi)}$$

$$\frac{- a\,fin.\lambda\,cos.(\lambda+\psi)^2)}{fin.(\lambda+\psi)}$$

ex hac aequatione statim sequitur fore $f = a\ fin.\ \lambda$, si ponatur $a\ fin.\ \lambda\ cos.\ (\lambda+\psi) = d\ fin.\ (\omega+\lambda+\psi)$; vel etiam hac aequatione diagonalis exprimitur

$$f = \frac{\sqrt{(a^2 fin.\lambda^2 - 2ad\,fin.\lambda\,cos.(\lambda+\psi)\,fin.(\omega+\lambda+\psi) +}}{fin.(\lambda+\psi)}$$

$$\frac{+ d^2 fin.(\omega+\lambda+\psi)^2)}{fin.(\lambda+\psi)}$$

Scholion III.

§. 74. Jam sex casuum in aequatione contentorum quinque sunt evoluti; qui ex aequatione satis expedite sequuntur, quorum tres utrique methodo et trigonometricae et tetragonometricae subsunt, ita tamen ut fere facilius trigonometrice quam tetragonometrice expediantur, alteri tres soli Tetragonometriae subsunt et trigonometrice non solvuntur, qui tunc obtinent, quando latus AB, angulus B, et diagonalis quaeritur, quorum duos, primum ac tertium per aequationes quadraticas satis concinnas breves ac utiles solvi. Secundus pro inveniendo angulo B ad aequationem sexti gradus vel altiorem ascendit, et consequenter aequatio ad praxin parum utilis evadit, cujus itaque studio brevitatis evolutionem omisi, verum suppeditant hi tres casus problemata in Geometria practica nova, utilia juxta ac satis pulchra.

Coroll. IV.

§. 75. Si anguli A et B ponantur ſimul ſumti aequales duobus rectis ut prodeat trapezium parallelarum baſium BC, AD, aequatio pro diagonali in hanc mutatur $f = \frac{a\, ſin.\, \lambda - d\, ſin.\, \omega}{o}$, et quia ex parallelismo conſtat, eſſe perpendicula aequalia inter ſe, fit $f = \frac{o}{o}$, quo indicatur problema fieri indeterminatum et manentibus iisdem lateribus AB, CD, diagonalem quemlibet valorem recipere poſſe, eſſe vero etiam perpendicula aequalia, ipſa aequatio monſtrat hoc modo $f.\ o = a\, ſin.\, \lambda - d\, ſin.\, \omega = o$ et conſequenter $a\, ſin.\, \lambda = d\, ſin.\, \omega$, ſed idem etiam indicat aequatio Coroll. I. cum fiat $a = \frac{d\ ſin.\ \omega}{ſin.\, \lambda}$.

Scholion IV.

§. 76. Quod problema in caſu laterum BC et AD parallelorum fiat indeterminatum, adhuc clarius apparet ex aequatione principali hactenus nihil immutata §. 68. in qua ſi ponatur $\lambda + \psi = \pi$, mutatur in hanc $a\, ſin.\, \lambda = ſin.\, \omega$, quam litera f non ingreditur, hoc ipſo indicatur diagonalem quemlibet valorem recipere poſſe, caeteris manentibus iisdem. Hic itaque monendi ſunt Geodaetae, ſi hoc in Geometria practica ſatis pulchrum et utile problema ad praxin transferre fuerit animus, ut ſibi caveant ab illo caſu, quo diagonalis evadit indeterminata, id quod ſemper praecavetur, utendo angulis A et B, quorum ſumma ſit major

vel minor duobus rectis, fic enim femper fcopo fuo potientur, nec, fi dicendum quod res eft, hic cafus Geodaetis erit magnopere pertimefcendus, cum omnium rariffimus exiftere debeat, quod dicti anguli fint duobus rectis aequales.

Coroll. V.

§. 77. Quod fi vero anguli B et C ponantur fimul fumti aequales duobus rectis, ut prodeat trapezium parallelarum bafium AB, CD aequatio pro diagonali mutatur in hanc

$$f = \frac{\sqrt{((a\,\mathit{fin.}\,\lambda\,\mathit{cos.}\,(\lambda+\psi) \pm d\,\mathit{fin.}\,\psi)^2 +}}{\mathit{fin.}\,(\lambda+\psi)}$$

$$\frac{+ a^2\,\mathit{fin.}\,\lambda^2\,\mathit{fin.}\,(\lambda+\psi^2))}{\mathit{fin.}\,(\lambda+\psi)}$$ vel etiam in hanc

$$f = \frac{\sqrt{(d^2\,\mathit{fin.}\,\psi^2 \pm 2ad\,\mathit{fin}\,\lambda\,\mathit{fin.}\,\psi\,\mathit{cos.}\,(\lambda+\psi) +}}{\mathit{fin.}\,(\lambda+\psi)}$$

$$\frac{+ a^2\,\mathit{fin.}\,\lambda^2)}{\mathit{fin.}\,(\lambda+\psi)}$$ fed aequatio pro latere AB Coroll. I. in hanc abit:

$$a = \frac{\pm d\,\mathit{fin.}\,\psi\,\mathit{cos.}\,(\lambda+\psi) \pm}{\mathit{fin.}\,\lambda.}$$

$$\frac{\pm \mathit{fin.}\,(\lambda+\psi)\sqrt{(f^2 - d^2\,\mathit{fin.}\,\psi^2)}}{\mathit{fin.}\,\lambda.}$$

Coroll. VI.

§. 78. Quod fi vero anguli A et B ponantur fimul fumti aequales uni recto, fiet diagonalis $f = \sqrt{(a^2\,\mathit{fin.}\,\lambda^2 + d^2\,\mathit{cos.}\,\omega^2)}$, et fi tribus rectis aequales ponantur fit $f = -\sqrt{(a^2\,\mathit{fin.}\,\lambda^2 + d^2\,\mathit{cos.}\,\omega^2)}$.

Signum

Signum negativum indicat, perpendicula jam cadere verſus ſiniſtram a latere AB, cum in priore caſu verſus dextram cecidiſſent, ſed in utroque caſu fit latus $AB = a = \frac{\surd(f^2 - d^2 \cos. \omega^2)}{\sin. \lambda}$

Sed ſi anguli B et C Sint uni recto, vel tribus rectis aequales, fit aequatio pro diagonali.

$$f = \frac{\surd(a^2 \sin. \lambda^2 \mp 2ad \sin. \lambda \cos. \psi \cos. (\lambda+\psi) +}{\sin. (\lambda+\psi)}$$

$\frac{+ d^2 \cos. \psi^2)}{\sin. (\lambda+\psi)}$, ubi ſignum negativum valet pro caſu priore, affirmativum pro poſteriore; Sed aequatio pro latere AB fit

$$a = \frac{d \cos. (\lambda+\psi) \cos. \psi \pm \sin. (\lambda+\psi)}{\sin. \lambda}$$

$$\frac{\surd(f^2 - d^2 \cos. \psi^2)}{\sin. \lambda.}$$

Coroll. VII.

§. 79. Quod ſi anguli diagonaliter oppoſiti A, C ſint ſimul ſumti duobus rectis aequales, ut prodeat trapezium circulo inſcriptibile, habetur pro diagonali haec aequatio:

$$f = \frac{\sin. \lambda \surd(a^2 + 2ad \cos. (\lambda+\psi) + d^2)}{\sin. (d+\psi)}$$

Si vero uni recto aequales ponantur, fit aequatio:

$$f = \frac{\surd(a^2 \sin. \lambda^2 - 2ad \sin. \lambda \cos. \lambda \cos. (\lambda+\psi) +}{\sin. (\lambda+\psi)}$$

$\frac{+ d^2 \cos. \lambda^2)}{\sin. (\lambda+\psi)}$. Si denique tribus rectis aequales ponantur, prodit eadem aequatio cum ſigno affirmati-

vo termini intermedii, sed in casu primo fit latus

$$AB = \frac{-d\,\text{fin.}\,\lambda\,\text{cos.}\,(\lambda+\psi) \pm}{\text{fin.}\,\lambda}$$

$$\frac{\pm\,\text{fin.}\,(\lambda+\psi)\,\sqrt{(f^2 - d^2\,\text{fin.}\,\lambda^2)}}{\text{fin.}\,\lambda} = a \text{ vel}$$

$$a = \frac{-d\,\text{cos.}\,(\lambda+\psi) \pm \text{fin.}\,(\lambda+\psi)\sqrt{(f^2 - d^2\,\text{fin.}\,\lambda^2)}}{\text{fin.}\,\lambda}.$$

In casu secundo et tertio habetur:

$$a = \frac{\pm\,d\,\text{cos.}\,\lambda\,\text{cos.}\,(\lambda+\psi) \pm}{\text{fin.}\,\lambda}$$

$$\frac{\pm\,\text{fin.}\,(\lambda+\psi)\,\sqrt{(f^2 - d^2\,\text{cos.}\,\lambda^2)}}{\text{fin.}\,\lambda},$$

ubi signum affirmativum valet pro casu priore, negativum pro posteriore.

Coroll. VIII.

§. 80. Si angulus A ponatur tribus rectis aequalis, cadente angulo A supra diagonalem BD, intra vel extra triangulum BCD, et prodeunte trapezio inverso, fit aequatio pro diagonali AC,

$$f = \frac{\sqrt{(a^2\,\text{fin.}\,\lambda^2 + 2ad\,\text{fin.}\,\lambda^2\,\text{cos.}\,(\omega+\lambda) +}}{\pm\,\text{cos.}\,\lambda}$$

$\frac{+ d^2\,\text{cos.}(\omega+\lambda)^2)}{\pm\,\text{cos.}\,\lambda}$ vel etiam quia signum radicale tam est affirmativum quam negativum, scribere licet.

$$f = \frac{\sqrt{(a^2\,\text{fin.}\,\lambda^2 + 2ad\,\text{fin.}\,\lambda^2\,\text{cos.}\,(\omega+\lambda) +}}{\text{cos.}\,\lambda}$$

$\frac{+ d^2\,\text{cos.}\,(\omega+\lambda)^2)}{\text{cos.}\,\lambda}$ Sed ponendo angulum B tribus rectis aequalem cadente vertice B ad dextram

tram diagonalis AC, intra vel extra triangulum ACD, prodeunte denuo trapezio inverſo, cujus diagonalis AC cadit extra figuram, habetur pro eadem diagonali haec aequatio:

$$f = \frac{\sqrt{(a^2 + 2ad\, fin.\psi\, cos.(\omega+\psi) + d^2 cos.(\omega+\psi)^2)}}{cos.\psi}.$$

Si denique angulus C ſit aequalis tribus rectis, cadente vertice C, infra diagonalem BD intra vel extra triangulum BAD, habetur pro diagonali AC haec aequatio:

$$f = \frac{\sqrt{(a^2 fin.\lambda^2 + 2ad\, fin.\lambda\, cos.(\lambda+\psi)^2 +}}{fin.(\lambda+\psi)}$$

$$\frac{+ d^2 cos.(\lambda+\psi)^2)}{fin.(\lambda+\psi)}.$$ In caſu primo aequatio primi Corollarii in hanc abit: $a = -d\, cos.(\omega+\lambda) \pm \pm cot.\lambda \sqrt{(f^2 - d^2\ cos.\ (\omega+\lambda^2))}$; in caſu ſecundo habetur $a = d\, fin.\psi\, cos.(\omega+\psi) \pm \pm cos.\psi \sqrt{(f^2 - d^2 cos.(\omega+\psi)^2)}$ denique in caſu tertio habetur haec aequatio:

$$a = \frac{-d\ cos.\ (\lambda+\psi)^2 \pm fin.\ (\lambda+\psi)}{fin.\lambda.}$$

$$\frac{\sqrt{(f^2 - d^2 cos.(\lambda+\psi)))}}{fin.\lambda.}$$

Scholion V.

§. 81. Si loco laterum AB, CD, quaeſtionem ingrediantur latera reliqua $BC = b$, et $AD = c$, haud abſimiliter ad aequationem ſimilem pervenio hoc modo. 1) In triangulo ABC eſt $fin.\ CAB = \frac{b\ fin.\ \lambda}{f}$ et $cos.\ CAB = = \frac{\sqrt{(f^2 - b^2 fin.\lambda^2)}}{f}$, conſequenter

fin.

$\mathit{sin.}\ ACB = \frac{\mathit{sin.}\ \lambda \surd (f^2 - b^2 \mathit{sin.}\ \lambda^2) + b\ \mathit{sin.}\ \lambda\ \mathit{cos.}\ \lambda}{f}$

nec non $\mathit{cos.}\ ACB = \frac{-\mathit{cos.}\ \lambda \surd (f^2 - b^2 \mathit{sin.}\ \lambda) + + b\ \mathit{sin.}\ \lambda^2}{f}$. 2) Hinc et ex ſinu et coſinu anguli C in triangulo ADC invenio

$\mathit{sin.}\ ACD = \frac{- \mathit{sin.} (\omega + \lambda \surd (f^2 - b^2 \mathit{sin.}\ \lambda^2) - - b\ \mathit{sin.}\ \lambda\ \mathit{cos.} (\omega + \lambda)}{f}$ et pervenio ad hanc analogiam $AC : \mathit{sin.}\ D = AD : \mathit{sin.}\ ACD$. h. e.

$f : -\mathit{sin.} (\omega + \lambda + \psi) = c : \frac{-\mathit{sin.} (\omega + \lambda) \surd (f^2 - b^2 \mathit{sin.} \lambda^2) - - b\ \mathit{sin.}\ \lambda\ \mathit{cos.} (\lambda + \psi))}{f}$ et fit aequatio:

$c\ \mathit{sin.} (\omega + \lambda + \psi) = \mathit{sin.} (\lambda + \omega) \surd (f^2 - b^2 \mathit{sin.}\ \lambda^2) + + b\ \mathit{sin.}\ \lambda\ \mathit{cos.} (\omega + \lambda)$ cujus igitur forma, cum forma prioris prorſus convenit, et haec ex illa prodit, c pro d, b pro a, ω pro ψ ſubſtituendo, ut jam pateat recte ſtatuiſſe Clariſſ. Lambert, hos caſus a ſe invicem analytice conſideratos, formaliter non differre.

Coroll. IX.

§. 82. Factis igitur debite ſubſtitutionibus, ſequitur reſolutiones ſupra datas Coroll. 1, 2, 3. in caſu laterum datorum oppoſitorum quaeſtionem ingredientium AB, CD etiam valere in hoc altero caſu, quo oppoſita latera AD, BC, quaeſtionem ingrediuntur.

Schol.

Scholion VI.

§. 83. Si loco lateris DC quaeſtionem ingrediatur latus $AD=c$, et ad aequationem perveniatur, erit ut ſupra $\text{ſin.}\, ACB = \frac{a\,\text{ſin.}\,\lambda}{f}$ et $\cos.\, ACB = \frac{\sqrt{(f^2 - a^2\,\text{ſin.}\,\lambda^2)}}{f}$ ex quibus et ex ſinu coſinuque anguli C invenio $\text{ſin.}\, ACD = \frac{\text{ſin.}\,\omega \sqrt{(f^2 - a^2\,\text{ſin.}\,\lambda^2)} - a\,\text{ſin.}\,\lambda \cos.\,\omega}{f}$, et in triangulo ADC pervenio ad hanc analogiam: $AC : \text{ſin.}\, D = AD : \text{ſin.}\, ACD$ hoc eſt, $f : -\text{ſin.}\,(\omega+\lambda+\psi) = c : \frac{\text{ſin.}\,\omega \sqrt{(f^2 - a^2\,\text{ſin.}\,\lambda^2)} - a\,\text{ſin.}\,\lambda \cos.\,\omega}{f}$ et conſequenter habetur $c\,\text{ſin.}\,(\omega+\lambda+\psi) = a\,\text{ſin.}\,\lambda \cos.\,\omega - \text{ſin.}\,\omega \sqrt{(f^2 - a^2\,\text{ſin.}\,\lambda^2)}$. Denique ſi quaeſtionem ingrediatur latus BC, loco lateris AB, pervenio ad hanc aequationem: $d\,\text{ſin.}\,(\omega+\lambda+\psi) = b\,\text{ſin.}\,\lambda \cos.\,\psi - \text{ſin.}\,\psi \sqrt{(f^2 - b^2\,\text{ſin.}\,\lambda^2)}$, cujus forma cum priori prorſus convenit. At vero ſi dicendum quod res eſt, hae aequationes, ad quas pervenitur ſumendis lateribus contiguis, differunt aliquid formaliter a prioribus, ad quas ſumendis oppoſitis pervenitur, cum in his in utroque termino dextri membri adſint ſinus et coſinus anguli unius; in illis vero ſinus et coſinus angulorum duorum ſimul ſumtorum; conſequenter hi duo caſus unum conſtituere cenſendi non ſunt, analytice et formaliter aliquid diverſum a caſu laterum oppoſitorum, id quod Cl. Lam-

Lambert non animadvertit, neque ego ſolam figuram intuendo ſupra in §. 15, 16, animadvertere potui, neque nunc animadvertiſſem, niſi aequationes ipſo facto formaſſem.

Coroll. X.

§. 84. Si ad abbreviandum ponatur $\mathit{ſin.}\,\omega = m$, $\cos.\,\omega = n$, $\mathit{ſin.}(\omega+\psi) = p$, $\cos.(\omega+\psi) = q$, facta ſeparatione ſinuum, ſubſtitutione et transpoſitione, poſitoque $\mathit{tang.}\,\lambda = t$, habetur $(f^2 m^2 - a^2 + 2acnq + c^2 q^2)\, t^2 + 2cp\,(an - c'q)\, t = c^2 p^2$, ex qua, operatione abſoluta, obtinetur $t = cp(cq - an \pm \surd((f^2-a^2)m^2 + 2c^2q^2)) = \mathit{tang.}\,\lambda$.

Scholion VII.

§. 85. Valet haec ſolutio non modo pro caſu laterum contiguorum AB, AD; ſed etiam contiguorum BC, CD, ſiquidem litteris c, d, ω, ſubſtituantur reſpective litterae d, b, ψ, cum ſic aequatio prior in poſteriorem abeat. Hi autem caſus laterum contiguorum hanc prae caſibus laterum oppoſitorum habent praerogativam, quod pro inveniendo angulo B, aequationem hanc ſatis utilem exhibeant, quam illi utilem dare non poterant. Caeteris ex his aequationibus evolvendis, ſtudio brevitatis ſuperſedeo.

Coroll. XI.

§. 86. Si angulus A ponatur aequalis duobus rectis, AB et AD in diagonalem BD incidentibus, et trapezio in triangulum BCD abeunte, habetur pro diagonali haec aequatio:

$f =$

$$f = \frac{\sqrt{(a^2 \mathit{sin}.\lambda^4 + (d\,\mathit{sin}.(\omega+\lambda) - a\,\mathit{sin}.\lambda\,\mathit{cos}.\lambda)^2)}}{\mathit{sin}.\lambda}$$

vel etiam $f = \frac{\sqrt{(a^2 \mathit{sin}.\lambda^2 - 2a\,d\,\mathit{sin}.\lambda\,\mathit{cos}.\lambda\,\mathit{sin}.(\omega+\lambda) +}}{\mathit{sin}.\lambda}$
$\frac{+ d^2\,\mathit{sin}.(\omega+\lambda)^2)}{\mathit{sin}.\lambda}$. Ex aequatione Coroll. 1. sequitur fore per hanc positionem: latus $CD =$
$d = \frac{\mathit{sin}.\lambda \sqrt{(f^2 - a^2 \mathit{sin}.\lambda^2)} + a\,\mathit{sin}.\lambda\,\mathit{cos}.\lambda}{\mathit{sin}.(\omega+\lambda)}$, nec non
latus $AB = a = \frac{d\ \mathit{cos}.\ \lambda\ \mathit{sin}.\ (\omega+\lambda) \mp}{\mathit{sin}.\lambda}$
$\frac{\mp \mathit{sin}.\lambda \sqrt{(f^2 - d^2\,\mathit{sin}.(\omega+\lambda)^2)}}{\mathit{sin}.\lambda}$, vel etiam
$a = d\,\mathit{cot}.\lambda\,\mathit{sin}.(\omega+\lambda) \mp \sqrt{(f^2 - \lambda\,\mathit{sin}.(\omega+\lambda)^2)}$
denique habetur angulus ad diagonalem superior.

$$\omega = \mathit{ang.sin}.\frac{\mathit{sin}.\lambda \sqrt{(f^2 - a^2 \mathit{sin}.\lambda^2)} + a\,\mathit{sin}.\lambda\,\mathit{cos}.\lambda}{d} - \lambda.$$

Si angulus C ponatur aequalis duobus rectis, lateribus BC, CD, in diagonalem BD incidentibus et trapezio in triangulum ADB abeunte, aequatio pro diagonali in hanc mutatur:

$f = \frac{\sqrt{(a^2 \mathit{sin}.\lambda^2 \mathit{sin}.(\lambda+\psi)^2 + (d\,\mathit{sin}.(\lambda+\psi) +}}{\mathit{sin}.(\lambda+\psi)}$
$\frac{+ a\,\mathit{sin}.\lambda\,\mathit{cos}.(\lambda+\psi))^2)}{\mathit{sin}.(\lambda+\psi)}$ vel etiam in hanc
$f = \frac{\sqrt{(a^2 \mathit{sin}.\lambda^2 + 2a\,d\,\mathit{sin}.\lambda\,\mathit{cos}.(\lambda+\psi)\,\mathit{sin}.(\lambda+\psi) +}}{\mathit{sin}.(\lambda+\psi)}$
$\frac{+ d^2\,\mathit{sin}.(\lambda+\psi)^2)}{\mathit{sin}.(\lambda+\psi)}$. Per hanc positionem aequatio pro latere CD in hanc mutatur
$d =$

$$d = \frac{\sin.(\lambda+\psi)\sqrt{(f^2 - a\sin.\lambda^2)} - a\sin.\lambda\cos.(\lambda+\psi)}{\sin.(\lambda+\psi)}$$

vel etiam $d = -a\sin.\lambda\cot.(\lambda+\psi) - \sqrt{(f^2 a^2 \sin.\lambda^2)}$. Sed aequatio pro latere AB in hanc mutatur

$$a = \frac{-d\sin.(\lambda+\psi)\cos.(\lambda+\psi) \pm \sin.(\lambda+\psi)}{\sin.\lambda} \; \frac{\sqrt{(f^2 - \lambda^2 \sin.(\lambda+\psi)^2)}}{\sin.\lambda}$$

vel etiam

$$a = \frac{\sin.(\lambda+\psi)}{\sin.\lambda}\left(-\cos.(\lambda+\psi \pm \sqrt{(f^2 - d^2\sin.(\lambda+\psi)^2)}\right)$$

Denique pro angulo A habetur haec aequatio: $\tang.(\lambda+\psi) = a\sin.\lambda : (\sqrt{(f^2 - a^2\sin.\lambda^2)} - d)$, et angulus ipſe $\psi = ang.\,tang.\dfrac{a\sin.\lambda - \lambda}{\sqrt{(f^2 - a^2\sin.\lambda^2)} - d}$

Si denique angulus B duobus rectis aequalis ponatur, lateribus BA, BC in diagonalem AC incidentibus, et trapezio in triangulum ACD abeunte, aequatio primo inventa in hanc abit: $d\sin.(\omega+\psi) = f\sin.\psi$, unde haec ſolita analogia trigonometrica: $f : \sin.(\omega+\psi) = d : \sin.\psi$, quo igitur ipſo aequationes et operationes verificantur.

Scholion VIII.

§. 87. Per poſitionem aequalitatis anguli A cum duobus rectis, nova ratione ad triangula ſpectans ſolvitur problema, quod tum obtinet, cum recta AC, ab angulo C ad baſin quomodocunque ducta quaeritur, nam in triangulo totali BCD, dantur latus CD et AB, ſegmentum lateris BD, et omnes anguli. In triangulo partiali ABC non dantur niſi duo, nempe latus AB, et angulus

angulus adjacens B; in altero partiali ACD, neque dantur nisi duo, puta latus CD, et angulus D: et tamen hac methodo tetragonometrica omnia triangula solvuntur; licet id ipsum etiam aliis methodis praestetur. Alterum ad triangula spectans solveretur problema novum, quando quaeritur angulus B, siquidem solutio satis utilis commoda et generalis pro inveniendo angulo B ex aequatione primo inventa erui posit. Caeteri casus etiamsi nova quadam ratione tetragonometrice solvantur, tamen etiam trigonometrice solvi possunt; cum si quaeratur segmentum AB in triangulo totali, tria data habeantur; et st quaeratur AD in triangulo ABC, etiam tria habeantur. Positione aequalitatis anguli C, cum duobus rectis novum rursus ad triangula spectans solveretur problema, siquidem ut modo dixi solutio satis commoda et utilis pro angulo B generalis haberetur; in hoc enim casu in triangulo totali ABD, non dantur nisi duo, puta angulus A, et latus AB, in partiali ABC non habentur nisi duo, puta latus AB et latus AC, nec in partiali ACD dantur nisi duo, latus AC et CD, et tamen omnia triangula solverentur.

Scholion IX.

§. 88. In hoc problemate aequatio inventa valet in hypothesi figurae constructae, quando scilicet angulus B est acutus et angulus ACD etiam acutus, sed ita ut summa earundem excedat rectum. Quod si enim eorundem summa sit recto minor valet haec aequatio: $d\,\textit{sin.}(\omega+\lambda+\psi)=$ $= a\,\textit{sin.}\,\lambda \cos(\psi-\lambda) - \textit{sin.}(\psi-\lambda)\sqrt{}(f^2-a^2\,\textit{sin.}\,\lambda^2)$

ſiquidem angulus A ſit major anguli B; ſi vero angulus B excedat angulum A, valet haec: $d\,\sin.(\omega+\lambda+\psi) = \sin.(\lambda-\psi)\sqrt{(f^2-a^2\sin.\lambda^2)} + a\sin.(\lambda-\psi)$. Si angulus B ſit obtuſus et major angulo A, valet haec aequatio: $d\,\sin.(\omega+\lambda+\psi) = \sin.(\lambda-\psi)\sqrt{(f^2-a^2\sin\lambda^2)} + a\sin.\lambda\cos.(\lambda-\psi)$. Si vero A ſit major quam B habetur haec: $d\,\sin.(\omega+\lambda+\psi) = a\sin.\lambda\cos.(x-\lambda) - \sin.(\psi-\lambda)\sqrt{(f^2-a^2\sin.\lambda^2)}$. Generalis igitur aequatio pro omni trapezio directo eſt hujusmodi: $d\,\sin.(\omega+\lambda+\psi) = \sin.\left(\begin{smallmatrix}\lambda\pm\psi\\ \psi-\lambda\end{smallmatrix}\right)\sqrt{(f^2-a^2\sin.\lambda^2)} + a\sin.\lambda\cos.\left(\begin{smallmatrix}\lambda\pm\psi\\ \psi-\lambda\end{smallmatrix}\right)$ ubi me non monente apparet propter ambiguum ſignum radicale et ambiguam naturam coſinus fieri poſſe, ut uterque terminus etiam negative ſit accipiendus. Si trapezium ſit partialiter inverſum, ita ut totum triangulum ADC, cadat intra triangulum ABC, anguli autem aequationem ingredientes ſint acuti, valet haec aequatio: $d\,\sin.(\omega+\lambda+\psi) = \sin.(\lambda+\psi)\sqrt{(f^2-a^2\sin.\lambda^2)} + a\sin.\lambda\cos.(\lambda+\psi)$, quae nihil differt ab ea, quae in reſolutione fuit inventa. Si trapezium ſit totaliter inverſum cadente latere AD intra latus AB, et latere CD extra latus CB, prodit haec aequatio: $d\,\sin.(\lambda+\psi-\omega) = \sin.(\lambda+\psi)\sqrt{(f^2-a^2\sin\lambda^2)} + a\sin.\lambda\cos.(\lambda+\psi)$. Quod ſi vero AD cadat extra latus AB, et latus CD intra latus CB; exiſtente angulo B majore quam DAB, obtinet haec aequatio: $d\,\sin.(\omega+\lambda-\psi) = \sin.(\lambda-\psi)\sqrt{(f^2-a^2\sin.\lambda^2)} + a\sin.\lambda\cos.(\lambda-\psi)$ Si vero caeteris manentibus iisdem contingat, angulum DAB excedere angulum B, obtinet haec:

$d\,\sin.$

$d\ \mathit{fin.} (\omega+\lambda-\psi) = -\mathit{fin.} (\psi-\lambda) \sqrt{} (f^2 - a^2 \mathit{fin.} \lambda^2) + a\ \mathit{fin.} \lambda\ \mathit{cos.} (\psi-\lambda)$ Si totum triangulum ABC cadat intra triangulum ADC, exiſtente trapezio partialiter inverſo et angulo B majore ſimul ſumtis angulis BAD et BCD, habetur haec aequatio: $d\ \mathit{fin.} (\lambda - \omega - \psi) = \mathit{fin.} (\lambda - \psi) \sqrt{} (f^2 - a^2 \mathit{fin.} \lambda^2) + a\ \mathit{fin.} \lambda\ \mathit{cos.} (\lambda - \psi)$. Si vero ſumma angulorum BAD, et BCD excedat angulum B, prodit haec aequatio: $d\ \mathit{fin.} (\omega+\psi-\lambda) = -\mathit{fin.} (\psi-\lambda \sqrt{} (f^2 - a^2 \mathit{fin.} \lambda^2) + a\ \mathit{fin.} \lambda\ \mathit{cos.} (\psi-\lambda)$. Si angulus B ponatur obtuſus et caeteri anguli aequationem et figuram ingredientes, ſive acuti ſive obtuſi, cum ipſo in aequatione et figura combinentur, plures iſtis variationes in aequatione non occurrunt, exceptis iis, quae ex ambiguitate ſigni radicalis et coſinus oriuntur. Generaliſſima igitur aequatio erit hujusmodi:

$$d\ \mathit{fin.} (\pm \omega \pm \lambda \pm \psi) = \mathit{fin.} \begin{pmatrix} \lambda \pm \psi \\ \psi - \lambda \end{pmatrix} \sqrt{} (f^2 - a\ \mathit{fin.} \lambda^2) + a\ \mathit{fin.} \lambda\ \mathit{cos.} \begin{pmatrix} \lambda \pm \psi \\ \psi - \lambda \end{pmatrix}.$$

Problema VII.

§. 89. In figura quadrilatera $ABCD$, inter haec ſex: latus $AB=a$, $BC=b$, diagonalem $AC=f$, et angulos, $A=\psi$, $C=\omega$, $D=\varphi$; aequationem invenire Fig. VII.

Solutio.

1) Per nota principia Geometrica erit nomen derivativum anguli quarti $B=2\pi-\omega-\varphi-\psi$ et hinc $\mathit{cos.}\ B = \mathit{cos.} (\omega+\varphi+\psi)$.

2) Ab angulo C cadat in latus oppoſitum AB perpendicularis Cc.

E 2 3) Hinc

3) Hinc in triangulo ABC per Euclid. Elem. II. 11. pervenio ſtatim ad hanc aequationem $AC^2 = AB^2 + BC^2 - 2AB.Bc$, et ſymbolis ſubſtitutis habetur: $f^2 = a^2 + b^2 + 2ab\ cos.(\omega + \varphi + \psi)$. Q. E. I.

Coroll. I.

§. 90. Ex hac aequatione eſt diagonalis, $AC = f = \sqrt{(a^2 + b^2 \pm 2ab\ cos.(\omega + \varphi + \psi))}$; latus $AB = a = \pm b\ cos.(\omega + \varphi + \psi) \pm \sqrt{(f^2 - b^2\ fin.(\omega + \varphi + \psi)^2)}$, et latus $BC = b = \pm a\ cos.(\omega + \varphi + \psi) \pm \sqrt{(f^2 - a^2\ fin.(\omega + \varphi + \psi)^2)}$.

Coroll. II.

§. 91. Sed eſt angulus $A = \psi = ang.\ cos. \pm \frac{(a^2 + b^2 - f^2)}{2ab} - \omega - \varphi$ angulus $C = \omega = ang. \pm cos. \frac{(a^2 + b^2 + f^2)}{2ab} - \varphi - \psi$, denique angulus $D = \varphi = ang.\ cos. \frac{(a^2 + b^2 - f^2)}{2ab} - \omega - \psi$.

Coroll. III.

§. 92. Si ipſorum angulorum ſinus quaerantur, habentur hae aequationes pro ſingulis angulis: $fin.\psi = \frac{(f^2 - a^2 - b^2)\ fin.(\omega + \varphi) \pm}{2ab}$ $\frac{\pm\ cos.(\omega + \varphi)\sqrt{(4a^2b^2 - (a^2 + b^2 - f^2)^2)}}{2ab}$

$fin.\varphi =$

$$\text{fin. } \varphi = \frac{(f^2 - a^2 - b^2)\ \text{fin. } (\omega + \psi) \pm}{2ab}$$
$$\frac{\pm \cos. (\omega + \psi) \sqrt{(4a^2 b^2 - (a^2 + b^2 - f^2)^2)}}{2ab}$$
$$\text{fin. } \omega = \frac{(f^2 - a^2 - b^2)\ \text{fin. } (\varphi + \psi) \pm}{2ab}$$
$$\frac{\pm \cos. (\varphi + \psi) \sqrt{(4a^2 b^2 - (a^2 + b^2 - f^2)^2)}}{2ab}$$

Scholion I.

§. 93. Sex cafuum qui in aequatione implicantur nullus eft Tetiagonometriae proprius, fed utrique methodo et Trigonometricae et Tetragonometricae communes, conftructionem trigonometricam pro diagonali aequatio tetragonometrica fuppeditat, pro lateribus et angulis conftructiones et folutiones tam funt faciles et fimplices ut folutionibus trigonometricis parum vel nihil cedant.

Coroll. IV.

§. 94. Si anguli diagonaliter oppofiti fimul fumti fint aequales duobus rectis, fit $f^2 = a^2 + b^2 \pm 2ab \cos. \varphi$. Si adjacentes lateri infimo AD fimul fumti fint aequales duobus rectis, ut prodeat trapezium parallelarum bafium AB, CD, fit aequatio: $f^2 = a^2 + b^2 \pm 2ab \cos. \omega$. Si adjacentes lateri CD duobus rectis aequales, ut prodeat trapezium parallelarum bafium AD, CB, fit aequatio: $f^2 = a^2 \pm b^2 + 2ab \cos. \psi$.

Coroll. V.

§. 95. Si angulus A duobus rectis aequalis, abeunte quadrilatero in triangulum BCD, fit aequatio: $f^2 = a^2 + b^2 \pm 2ab\ cos.\ (\omega + \varphi)$. Si D duobus rectis aequalis abeunte trapezio in triangulum ABC fit: $f^2 = a^2 + b^2 \pm 2ab\ cos.\ (\omega + \psi)$; Si angulus C duobus rectis aequalis, abeunte trapezio in triangulum $B\ A\ D$, fit: $f^2 = a^2 + b^2 \pm 2ab\ cos.\ (\varphi + \psi)$.

Coroll. VI.

§. 96. Si anguli diagonaliter oppositi aequentur simul uni recto, habetur haec aequatio: $f^2 = a^2 + b^2 \pm 2ab\ sin.\ \varphi$; Si adjacentes lateri AD simul sumti uni recto aequales, fit aequatio: $f^2 = a^2 + b^2 \pm 2ab\ sin.\ \psi$.

Coroll. VII.

§. 97. Si diagonaliter oppositi simul sumti aequentur tribus rectis, habetur pro figura haec aequatio: $f^2 = a^2 + b^2 \pm 2ab\ sin.\ \varphi$; Si adjacentes lateri AD, aequales tribus rectis, fit aequatio: $f^2 = a^2 + b^2 \pm 2ab\ sin.\ \omega$; Si adjacentes lateri $C\ D$ aequales tribus rectis, fit: $f^2 = a^2 + b^2 + 2ab\ sin.\ \psi$.

Coroll. VIII.

§. 98. Si anguli A, D, C, ponantur sigillatim aequales tribus rectis, cadente in singulo casu vertice ultra diagonalem intra vel extra triangulum remotius, et prodeunte trapezio inverso, fit

in

in primo casu aequatio: $f^2 = a^2 + b^2 \pm 2ab\,\mathit{sin.}(\omega + \varphi)$
in secundo: $f^2 = a^2 + b^2 \pm 2ab\,\mathit{sin.}\,(\omega + \psi)$
in tertio: $f^2 = a^2 + b^2 \pm 2ab\,\mathit{sin.}\,(\varphi + \psi)$.

Scholion II.

§. 99. Aequatio hujus problematis pro trapezio directo generalis haec est: $f^2 = a^2 + b^2 \mp 2ab\,\mathit{cos.}\,(\omega + \varphi + \psi)$. Si trapezium sit partialiter inversum, ut totum triangulum ADC cadat intra triangulum ABC, valet haec aequatio: $f^2 = a^2 + b^2 - 2ab\,\mathit{cos.}\,(\varphi - \omega - \psi)$. Quod si latus AD cadat intra latus AB, latus autem CD extra latus CB, ut trapezium sit totaliter inversum, erit aequatio: $f^2 = a^2 + b^2 - 2ab\,\mathit{cos.}\,(\varphi + \omega - \psi)$; Si vero trapezium ita sit totaliter inversum, ut latus AD cadat extra latus AB, latus autem CD intra latus CB, habetur haec aequatio: $f^2 = a^2 + b^2 - 2ab\,\mathit{cos.}\,(\varphi + \psi - \omega)$. Quod si trapezium ita sit partialiter inversum, ut totum triangulum ABC cadat intra triangulum ADC; prodit aequatio in resolutione problematis inventa. Si angulus B obtusus ponatur, et caeteri cum ipso sive acuti sive obtusi quomodocunque combinentur, eaedem aequationes prodibunt, quare generalis aequatio erit hujusmodi: $f^2 = a^2 + b^2 + 2ab\,\mathit{cos.}\,(\varphi \pm \omega \pm \psi)$.



CAPUT IV.

Continens tria problemata secundae classis particularis, sub priore principali contentae.

Problema VIII.

Fig. VIII. 1. §. 100. In figura quadrilatera rectilinea ABC, inter haec sex latus $AB=a$, diagonalem $AC=f$, latus $AD=c$, atque tres angulos $A=\psi$, $ACB=\alpha$, et $ACD=\beta$; aequationem invenire.

Solutio.

1) In triangulo sinistro formo hanc analogiam: $AB : \text{sin.}\ ACB = AC : \text{sin.}\ B$ et substitutis symbolis $a : \text{sin.}\ \alpha = f : \text{sin.}\ B = \frac{f\ \text{sin.}\ \alpha}{a}$, consequenter $\text{cos.}\ B = \frac{\sqrt{(a^2 - f^2\ \text{sin.}\ \alpha^2)}}{a}$

2) Ex datis nominibus primitivis angulorum A, ACB, ACD, formo nomen derivativum sinus et cosinus summae eorundem, quorum prius $\text{sin.}\ (\alpha+\beta+\psi)$, posterius $\text{cos.}\ (\alpha+\beta+\psi)$.

3) Ex his, et ex sinu cosinuque anguli B, formo sinum summae trium angulorum, A, B, C, et obtineo $\text{sin.}\ (A+C+B) = \frac{-\text{sin.}\ (\alpha+\beta+\psi)}{a}$

$$\frac{\sqrt{(a^2-f^2\ \text{sin.}\ \alpha^2)} - f\ \text{sin.}\ \alpha\ \text{cos.}\ (\alpha+\beta+\psi)}{a}$$

4) Cum

4) Cum fit $fin.\ ((A+B+C+D)-(A+B+C)) = fin.\ (2\pi - (A+B+C)) = fin.\ D$, erit propter $fin.\ 2\pi = 0$, et $cos.\ 2\pi = 1$,

$$fin.\ D = \frac{fin.\ (\alpha+\beta+\psi) \surd (a^2 - f^2\, fin.\ \alpha^2) + f\, fin.\ \alpha\ cos.\ (\alpha+\beta+\psi)}{a}$$

et in triangulo ABC pervenio ad hanc analogiam. $AC : fin.\ D = AD : fin.\ ACD$, h. e.

$$f : \frac{fin.\ (\alpha+\beta+\psi) \surd (a^2 - f^2 fin.\ \alpha^2) + f\, fin.\ \alpha\ cos.\ (\alpha+\beta+\psi)}{a} = c : fin.\ \beta,$$

et consequenter habetur: $a\, f\, fin.\ \beta = c\, fin.\ (\alpha+\beta+\psi) \surd (a^2 - f^2\, fin.\ \alpha^2) + f c\, fin.\ \alpha.\ cos.\ (\alpha+\beta+\psi)$ Q. E. I.

Scholion I.

§. 101. Sinum fummae trium angulorum, A, B, C, in folutione fumfi negativum, quia in figura conftructa haec fumma excedit duos rectos, ac inter duos et tres rectos cadit. Sed prolixius ad eandem aequationem perveniffem et angulorum B, et ACB, formatis finibus et cofinibus, formando finum et cofinum anguli tertii CAB, et ex his, nec non finu cofinuque anguli A, conficiendo finum et cofinum anguli CAD, denique ex his, et ex finu cofinuque anguli ACD, componendo finum quaefitum anguli D. Sic enim prodiiffet

$$fin.\ D = \frac{- fin.\ (\alpha+\beta+\psi) \surd (a^2 - f^2\, fin.\ \alpha^2) - f\, fin.\ \alpha\ cos.\ (\alpha+\beta+\psi)}{a}$$

 et

et aequatio $a f \text{ fin.} \beta = - c \text{ fin.} (\alpha+\beta+\psi) \surd(\alpha^2 - f^2 \text{ fin.} \alpha^2) - c f \text{ fin.} \alpha \cos. (\alpha+\beta+\psi)$, quae fignis tantum a priori differt, quae tum ex natura finuum & cofinuum, tum etiam ex natura figni radicalis ambigua effe poffunt.

Scholion II.

§. 102. Altera folutio obtinetur formando in triangulo ABC hanc analogiam $AD : \text{fin.} ACD = AC : \text{fin.} D$. h. e. $c : \text{fin.} \beta = f : \text{fin.} D = \frac{f \text{ fin.} \beta}{c}$, et confequenter $\cos. D = \frac{\surd(c^2 - f^2 \text{ fin.} \beta)}{c}$, hinc erit finus fummae trium angulorum A, D, C, $= \text{fin.} (A+D+C) = \frac{- \text{fin.} (\alpha+\beta+\psi)}{c}$ $\frac{\surd(c^2 - f^2 \text{ fin.} \beta^2) - f \text{ fin.} \beta \cos. (\alpha+\beta+\psi)}{c}$, et confequenter $\text{fin.} B = \frac{\text{fin.} (\alpha+\beta+\psi)}{c}$ $\frac{\surd(c^2 - f^2 \text{ fin.} \beta^2) + f \text{ fin.} \beta \cos. (\alpha+\beta+\psi)}{c}$; et in triangulo ACD pervenietur ad hanc analogiam. $AC : \text{fin.} B = AB : \text{fin.} ACB$. h. e. $f : \text{fin.} \frac{(\alpha+\beta+\psi) \surd(c^2 - f^2 \text{ fin.} \beta^2) +}{c}$ $\frac{+ f \text{ fin.} \beta \cos. (\alpha+\beta+\psi)}{c} = a : \text{fin.} \alpha$ et ad hanc aequationem: $c f \text{ fin.} \alpha = a \text{ fin.} (\alpha+\beta+\psi) \surd(c^2 - f^2 \text{ fin.} \beta^2) + a f \text{ fin.} \beta \cos. (\alpha+\beta+\psi)$ Si ex aequatione priore radix quadratica quaeratur;

tur; $c f \text{ſin.}\, \alpha$ hujus dextrum membrum tanquam radix obtinetur, quoad omnia cum ipſo conveniens, niſi quod per naturam radicis quadraticae terminum ſigno radicali affectum, duplici ſigno affici contingat, et affirmativo ſigno adhibito, prodeat dextrum membrum. Quod ſi vero ex aequatione eadem prout in ſcholio praecedenti inventa fuit cum ſignis negativis quaeratur $c f \text{ſin.}\, \alpha$, idem obtinetur, niſi quod terminus dextimus prodeat negativus, id quod naturae coſinus non repugnat. Quod ſi ergo ex aequatione hac poſteriore per extractionem radicis quadraticae quaeram $a f \text{ſin.}\, \beta$, obtinetur aequatio primo inventa. Cum igitur harum aequationum una ſit radix alterius, pro realiter diverſis habendae non ſunt, ſed formis quibus primo prodierunt tantum diſcrepant.

Coroll. I.

§. 103. Ex aequatione obtinetur facillime latus $A\,D == c ==$

$$\frac{f\,c\,\text{ſin.}\,\alpha}{\text{ſin.}(\alpha+\beta+\psi)\sqrt{}(a^2-f^2\,\text{ſin.}\,\alpha^2)+f\,\text{ſin.}\,\alpha\cos.(\alpha+\beta+\varphi)}$$

Sed ex poſteriore h. e. ex priore in formam poſteriorem transmutata habetur latus $A\,B == a ==$ $= f\,c\,\text{ſin.}\,\alpha : (\text{ſin.}\,(\alpha+\beta+\psi)\sqrt{}(c^2-f^2\,\text{ſin.}\,\beta^2)+ f\,\text{ſin.}\,\beta\cos.\,(\alpha+\beta+\psi))$.

Coroll. II.

§. 104. Ex poſteriore aequatione, ſive ex priore in formam poſterioris transmutata, obtinetur pro angulo $A\;C\;B$ haec aequatio:

tang.

$tang.\ \alpha =$
$$\frac{af\,fin.\beta\,cos.(\beta+\psi)+a\,fin.(\beta+\psi)\sqrt{(c^2-f^2\,fin.\beta^2)}}{cf+af\,fin.\beta\,fin.(\beta+\psi)-a\,cos.(\beta+\psi)\sqrt{(c^2-f^2\,fin.\beta^2)}}$$
etiam haec paullo fimplicior divendo per $cos.(\beta+\psi)$
$tang.\ \alpha =$
$$\frac{af\,fin.\beta+a\,tang.(\beta+\psi)\sqrt{(c^2-f^2\,fin.\beta^2)}}{fc\,fec.(\beta+\psi)+af\,fin.\beta\,tang.(\beta+\psi)-a\sqrt{(c^2-f^2\,fin.\beta^2)}}$$
Sed ex aequatione priore, vel etiam pofteriore in formam prioris transfufa, pro angulo $A\ C\ D$ habetur haec aequatio:
$tang.\ \beta =$
$$\frac{fc\,fin.\alpha\,cos.(\alpha+\psi)+c\,fin.(\alpha+\psi)\sqrt{(a^2-f^2\,fin.\alpha^2)}}{fa+fc\,fin.\alpha\,fin.(\alpha+\psi)-c\,cos.(\alpha+\psi)\sqrt{(a^2-f^2\,fin.\alpha^2)}}$$
vel etiam haec altera paullo fimplicior:
$tang.\ \beta =$
$$\frac{fc\,fin.\alpha+c\,tang.(\alpha+\psi)\sqrt{(a^2-f^2\,fin.\alpha^2)}}{fa\,fec.(\alpha+\psi)+fc\,fin.\alpha\,tang.(\alpha+\psi)-c\sqrt{(a^2-f^2\,fin.\alpha^2)}}$$

Coroll. III.

§. 105. Pofito vel abbreviandum $fin.\ \beta = m$, $fin.\psi = x$, $cos.\psi = \sqrt{(1-x^2)}$, $fc\,fin.\alpha\,fin.(\alpha+\beta) + c\ cos.(\alpha+\beta)\sqrt{(a^2-f^2\ fin.\alpha^2)} = A$ et $fc\,fin.\alpha\,cos.(\alpha+\beta)+c\,fin.(\alpha+\beta)\sqrt{(a^2-f^2\,fin.\alpha^2)} = B$, factis fubftitutionibus obtinetur haec aequatio: $amf - Ax = B\sqrt{(1-x^2)}$, ex qua operatione abfoluta habetur pro angulo A.
$$x = \frac{Aamf \pm B\sqrt{(A^2+B^2-a^2m^2f^2)}}{A^2+B^2} = fin.\ \phi.$$
Sed aequationem priorem quadrando, terminum $a^2f^2 fin.\beta^2$ multiplicando per $fin.(\alpha+\beta+\phi^2)+cos.(\alpha+\beta+\phi)^2 = 1$, et radicem quadraticam extra-

extrahendo, operatione abſoluta pervenitur ad hanc aequationem pro angulo A

$$tang.\ (\alpha+\beta+\psi) = \frac{c^2 f fin.\alpha \sqrt{(a^2-f^2 fin.\alpha^2)} \pm a^2 f fin.\beta \sqrt{(c^2-f^2 fin.\beta^2)}}{(af fin.\beta + c\sqrt{(a^2-f^2 fin.\alpha^2)})af fin.\beta - c\sqrt{(a^2-f^2 fin.\alpha^2)})}$$

et conſequenter ipſe angulus A.

$$\psi = ang.\ tang.\ \frac{(c^2 f fin.\alpha \sqrt{(a^2-f^2 fin.\alpha^2)} \pm a^2 f fin.\beta \sqrt{(c^2-f^2 fin.\beta^2)}}{(af fin.\beta + c\sqrt{(a^2-f^2 fin.\alpha^2)})(af fin.\beta - c\sqrt{(a^2-f^2 fin.\alpha^2)})} - \alpha - \beta$$

Coroll. IV.

§. 106. Ex utraque aequatione idem pro diagonali valor obtinetur nimirum, hic ex priore:

$$f = \frac{a\ c\ fin.\ (\alpha+\beta+\psi)}{\sqrt{(a fin.\beta - c fin.\alpha cos(\alpha+\beta+\psi))^2 + c^2 fin.\alpha^2 fin.(\alpha+\beta+\psi)^2)}}$$

ex poſteriore hic:

$$f = \frac{a\ c\ fin.\ (\alpha+\beta+\psi)}{\sqrt{(c fin.\alpha - a fin.\beta cos.(\alpha+\beta+\psi))^2 + a^2 fin.\beta^2 fin.(\alpha+\beta+\psi)^2)}}$$

quarum expreſſionum una in alteram facile mutatur, ut ex utraque prodeat:

$$f = \frac{a\ c\ fin.\ (\alpha+\beta+\psi)}{\sqrt{(c^2 fin.\alpha^2 - 2ac fin.\alpha fin.\beta cos.(\alpha+\beta+\psi) + a^2 fin.\beta^2)}}$$

Scholion III.

§. 107. Ita ergo evoluti ſunt ſex illi caſus qui in aequatione tetragonometrica continentur, horum vero quinque non ſunt Tetragonometriae proprii, ſed utrique methodo et tetragonometricae et trigonometricae ſubſunt, quos tamen ideo evolvi, quia ex aequatione ſine labore eruuntur, facilius tamen et expeditius fere trigonometrice ſolvun-

ſolvuntur, et per Geometriam elementarem. Solus ille caſus, quo diagonalis quaeritur, eſt Tetragonometriae proprius, neque enim trigonometrice ſolvitur, continet vero hic caſus problema in Geometria practica novum, utile pulchrum et egregium.

Coroll. V.

§. 108. Si anguli diagonaliter oppoſiti ponantur ſimul ſumti aequales duobus rectis, aequatio utraque mutatur in hanc: $\alpha\ fin.\ \beta = - c\ fin.\ \alpha$, ubi ſigni negativi ratio habenda non eſt, cum nihil indicet niſi ſinus angulorum ACB et ACD cadere in partes oppoſitas reſpectu diagonalis, uti revera in figura ſe res habet, unde prodit haec analogia: $a : c = fin.\ \alpha : fin.\ \beta$, quae ſuppeditat hoc Theorema:

Si in trapezio circulo inſcriptibili ducatur diagonalis; erunt latera contigua inter ſe ut ſinus angulorum ipſis oppoſitorum. Eſt enim in utroque triangulo $AB : fin.\ ACB = AC : fin.\ B$, et $AD : fin.\ ACD = AC : fin.\ D$; ſed ex hypotheſi ſequitur, reliquos etiam angulos diagonaliter oppoſitos B et D eſſe ſimul ſumtos duobus rectis aequales. Cum igitur ſit unus alterius complementum ad ſemicirculum, uterque eundem habet ſinum, et hinc ſequitur eſſe $AB : fin.\ ACB = AD : fin.\ ACD$. Eodem modo ſequitur eſſe $AD : fin.\ ABD = DC : fin.\ DBC$, unde porro ſequitur demonſtratio Theorematis §. 57. allati, eſſe nimirum $AC : fin.\ B = BD : fin.\ E$.

Coroll.

Coroll. VI.

§. 109. Si angulus A ponatur aequalis duobus rectis, lateribus datis AB, AD, in diagonalem BD incidentibus et trapezio in triangulum BCD mutato; aequatio pro diagonali in hanc abit:

$$f = \frac{ac \text{ fin. } (\alpha+\beta)}{\sqrt{((a \text{ fin. } \beta \pm c \text{ fin. } \alpha \cos.(\alpha+\beta))^2 + c^2 \text{ fin. } \alpha^2 \text{ fin. } (\alpha+\beta)^2)}},$$

vel etiam in hanc alteram:

$$f = \frac{ac \text{ fin. } (\alpha+\beta)}{\sqrt{(a^2 \text{ fin } \beta^2 \pm 2ac \text{ fin. } \alpha \text{ fin. } \beta \cos.(\alpha+\beta) + c^2 \text{ fin. } \alpha^2)}}.$$

Si vero angulus C ponatur aequalis duobus rectis, lateribus BC, CD in diagonalem BD incidentibus et trapezio iterum in triangulum BAD abeunte, obtinetur haec aequatio pro diagonali:

$$f = \frac{ac \text{ fin. } \psi}{\sqrt{(c \text{ fin. } \alpha \pm a \text{ fin. } \beta \cos. \psi)^2 + a^2 \text{ fin. } \beta^2 \text{ fin. } \psi^2)}}$$

vel etiam haec altera:

$$f = \frac{a \text{ fin. } \psi}{\sqrt{(c^2 \text{ fin. } \alpha^2 \pm 2ac \text{ fin. } \alpha \text{ fin. } \beta \cos. \psi + a^2 \text{ fin. } \beta^2)}}$$

vel etiam unum angulorum α et β eliminando habetur $f = \frac{ac \text{ fin. } \psi}{\text{fin. } \alpha \sqrt{(c^2 \pm 2ac \cos. \psi + a^2)}}$ vel etiam $f = \frac{ac \text{ fin. } \psi}{\text{fin. } \beta \sqrt{(c^2 \pm 2ac \cos. \psi + a^2)}}$

Scholion IV.

§. 110. Prior positio de triangulo BCD, tetragonometrice solvit hoc problema: Ex dato latere toto BD et partibus ejus EB, ED, et angulis ECB, ECD invenire rectam EC, et resolvere

folvere triangula ECB, ECD. Pofterior pofitio de triangulo BAD folvit hoc problema: Ex datis lateribus et angulo intercepto, nec non angulis AEB, AED, invenire rectam AE, et refolvere triangula. Prius problema aliis methodis non, hoc pofterius vulgari etiam methodo folvitur.

Coroll. VII.

Fig. VIII. 2. §. 111. Si angulus A ponatur tribus rectis aequalis, lateribus datis AB, AD fupra diagonalem BD cadentibus, prodibit pro diagonali haec aequatio:

$$f = \frac{ac\ \text{cof.}\ (\alpha+\beta)}{\sqrt{((a\,\text{fin.}\beta + c\,\text{fin.}\alpha\,\text{fin.}(\alpha+\beta))^2 + c^2\,\text{fin.}\alpha^2\,\text{cof.}(\alpha+\beta)^2)}}$$

vel etiam haec:

$$f = \frac{ac\ \text{cof.}\ (\alpha+\beta)}{\sqrt{(c^2\,\text{fin.}\,\alpha + 2ac\,\text{fin.}\,\alpha\,\text{fin.}\,\beta\,\text{fin.}\,(\alpha+\beta) + a^2\,\text{fin.}\,\beta^2)}}$$

Quod fi vero angulus C ponatur aequalis tribus rectis, lateribus CB, CD, infra diagonalem BD cadentibus; prodibit pro diagonali haec aequatio:

$$f = \frac{ac\ \text{cof.}\ \psi}{\sqrt{((a\,\text{fin.}\,\beta \pm c\,\text{fin.}\,\alpha\,\text{fin.}\,\psi)^2 + c^2\,\text{fin.}\,\alpha^2\,\text{cof.}\,\psi^2)}}$$

vel etiam

$$f = \frac{ac\ \text{cof.}\ \psi}{\sqrt{(a^2\,\text{fin.}\,\beta^2 \pm 2ac\,\text{fin.}\,\alpha\,\text{fin.}\,\beta\,\text{fin.}\,\psi + c^2\,\text{fin.}\,\alpha^2)}}$$

vel etiam alterutrum angulum α, β eliminando habetur

$$f = \frac{ac\ \text{cof.}\ \psi}{\sqrt{(a^2\,\text{fin.}\,\beta^2 \pm 2ac\,\text{fin.}\,\beta\,\text{cof.}\,\beta\,\text{fin.}\,\psi + c^2\,\text{cof.}\,\beta^2)}}$$

et

et dividendo per *cos.* β^2 aequatio in hanc mutatur:

$$f = \frac{ac\ cos.\psi\ sec.\beta^2}{\sqrt{(a^2 \pm 2ac\ sin.\psi\ tang.\beta + \alpha^2\ tang.\beta^2)}}$$

vel etiam eliminando β et per *cos.* α dividendo habetur.

$$f = \frac{ac\ cos.\varphi\ sec.\alpha.}{\sqrt{(a^2 \pm 2ac\ sin.\psi\ tang.\alpha + c^2\ tang.\alpha^2)}}.$$

Scholion V.

§. 112. In his duobus casibus prodit trapezium, cujus una diagonalis cadit extra figuram, quod ex parte inversum appello, sed totaliter inversum trapezium voco, cujus utraque diagonalis cadit extra figuram, his enim trapeziis ob figuram insolitam et paradoxam, peculiare nomen imponere operae praetium videtur, etiam si hactenus id a Geometris factum non fuerit; directum ergo trapezium est, cujus utraque diagonalis cadit intra figuram. Fig.VIII. 3.

Coroll. VIII.

§. 113. Quod si unus angulorum ACB, ACD respectu sui negativus ponatur, hoc est, si angulus $ACD = \beta$, qui in figura primo proposita cadit ad dextram diagonalis, cadat ad sinistram ejusdem, vel e contra, angulus ACB ad sinistram cadens, ad dextram sit positus, prodibit in utroque casu trapezium inversum, et pro casu priore habetur pro diagonali extra figuram cadente:

$$f = \frac{ac\ sin.(\alpha+\beta+\psi)}{\sqrt{((\pm c\ sin.\alpha\ cos.(\alpha-\beta+\psi) - a\ sin.\beta)^2 + c^2\ sin.\alpha^2\ sin.(\alpha-\beta+\psi^2))}}$$

F vel

vel etiam:

$$f = \frac{a\,c\,\text{ſin.}(\alpha - \beta + \psi)}{\sqrt{(c^2\,\text{ſin.}\,\alpha^2 \pm 2ca\,\text{ſin.}\,\alpha\,\text{ſin.}\,\beta\,\text{cos.}(\alpha - \beta + \psi) + a^2\,\text{ſin.}\,\beta^2)}}$$

Sed pro caſu poſteriore habentur hae plane ſimiles:

$$f = \frac{a\,c\,\text{ſin.}\,(\beta - \alpha + \psi)}{\sqrt{((\pm a\,\text{ſin.}\,\beta\,\text{cos.}(\beta - \alpha + \psi) - c\,\text{ſin.}\,\alpha)^2 + a^2\,\text{ſin.}\,\beta^2\,\text{ſin.}\,\beta - \alpha + \psi^2))}}$$

vel etiam:

$$f = \frac{a\,c\,\text{ſin.}\,(\beta - \alpha + \psi)}{\sqrt{(c^2\,\text{ſin.}\,\alpha \pm 2ac\,\text{ſin.}\,\alpha\,\text{ſin.}\,\beta\,\text{cos.}(\beta - \alpha + \psi) + a^2\,\text{ſin.}\,\beta^2))}}$$

Coroll. IX.

§. 114. Quod ſi in his poſitionibus praeterea ponantur anguli α et β aequales inter ſe, latus CD cadet ſuper latus BC et ſic diagonali una extra figuram cadente, altera neque intra neque extra figuram cadet, ſed incidet in latus BC, ut partem ejus conſtituat CD, et ſic nova prodiret ſpecies trapezii, niſi majore jure haberetur pro triangulo, uno latere producto, in hoc autem caſu habetur pro diagonali extra figuram haec aequatio:
$f = ac\,\text{ſin.}\,\psi : \text{ſin.}\,\alpha \sqrt{(c^2 \pm 2ac\,\text{cos.}\,\psi + a^2)}$ vel $f = ac\,\text{ſin.}\,\psi : \text{ſin.}\,\beta \sqrt{(c^2 \pm 2ac\,\text{cos.}\,\psi + a^2)}$. Si vero latus AD longius ſit, quam ut totum intra figuram cadere poſſit, vel etiam, quod eodem recidit, angulus $ACD = \beta$ major angulo $ACB = \alpha$, prodit trapezium totaliter inverſum.

Coroll. X.

§. 115. Si anguli diagonaliter oppoſiti A, C, ponantur ſimul ſumti uni recto aequales, ita ut

ut reliqui oppofiti B, D, fiant fimul tribus rectis aequales, fiet diagonalis:

$f = \frac{ac}{\sqrt{(c^2 \text{fin.} \alpha^2 + a^2 \text{fin.} \beta^2)}}$. Si vero anguli ACB, ACD fimul fumti fint uni recto aequales, five angulus C rectus; prodibit haec aequatio: $f = ac \cos. \psi \sec. \alpha^2 : \sqrt{(c^2 \tang. \alpha \pm 2ac \tang. \alpha \text{fin.} \psi + a^2)}$ vel $f = ac \cos. \psi \sec. \beta : \sqrt{(c^2 \pm 2ac \text{fin.} \psi \tang. \beta + a^2 \tang. \beta^2)}$. Si denique angulus ACB vel ACD evanefcere ponatur, trapezio in triangulum abeunte, prodit in uno cafu $f \text{fin.} \beta = c \text{fin.} (\beta + \psi)$; in altero: $f \text{fin.} \alpha = a \text{fin.} (\alpha + \psi)$, et confequenter prodeunt analogiae trigonometricae in utroque triangulo, $f : c = \text{fin.} (\beta + \psi) : \text{fin.} \beta$, vel $c : \text{fin.} \beta = f : \text{fin.} (\beta + \psi)$, et $f : a = \text{fin.} (\alpha + \psi) : \text{fin.} \alpha$ vel $a : \text{fin.} \alpha = f : \text{fin.} (\alpha + \psi)$, quo ipfo aequationes et operationes verificantur:

Coroll. XI.

§. 116. Quod fi in aequatione priore ponatur $a = f$; in hanc mutatur: $f \text{fin.} \beta = \pm c \text{fin.} (\alpha + \beta + \psi) \cos. \alpha + c \cos. (\alpha + \beta + \psi) \text{fin.} \alpha$. Haec aequatio figno affirmativo adhibito in hanc abit: $f \text{fin.} \beta = + c \text{fin.} (2\alpha + \beta + \psi)$; verum adhibito negativo in hanc: $f \text{fin.} \beta = c \text{fin.} (\beta + \psi)$. Sed ponendo $c = f$, aequatio pofterior mutatur in hanc: $f \text{fin.} \alpha = \pm a \text{fin.} (\alpha + \beta + \psi) \cos. \beta + a \text{fin.} \beta \cos. (\alpha + \beta + \psi)$ quae figno affirmativo adhibito in hanc abit: $f \text{fin.} \alpha = a \text{fin.} (\alpha + 2\beta + \psi)$, fed negativo in hanc: $f \text{fin.} \alpha = a \text{fin.} (\alpha + \psi)$; aequationes fignis affirmativis adhibitis revera locum habent in trapezio directo, pofteriores

 locum

locum non nisi in uno casu, quo angulus $ACD = \alpha$ in altero angulo $ACD = \beta$ evanescat, et consequenter trapezium in triangulum abeat. Per positionem priorem statim sequitur esse diagonalem sive latus $AB = a = f$ $= \frac{-c\,fin.\,(2\alpha+\beta+\psi)}{fin.\,\beta}$, et latus $AD = c$ $= -f\,fin.\,\beta : fin.\,(2\alpha+\beta+\psi)$, et cum summa angulorum sit, $2\alpha+\beta+\psi = ang.\,fin.\frac{-f\,fin.\,\beta}{c} - \frac{\beta}{2} - \frac{\psi}{2}$ et angulus $\psi = ang.\,fin.\frac{-f\,fin.\,\beta}{c} - 2\alpha - \beta$; denique pro angulo ACD habetur $tang.\,\beta = \frac{-c\,fin.\,(2\alpha+\psi)}{f+c\,cos.\,(2\alpha+\psi)}$

Ex aequatione posteriore in casu evanescentis anguli ACB nihil sequitur, nisi solita in triangulo ACD analogia trigonometrica $f : fin.\,(\beta+\psi) = c : fin.\,\beta$. At vero per positionem posteriorem statim sequitur esse latus $AD = AC = f = \frac{-a\,fin.\,(\alpha+2\beta+\psi)}{fin.\,\alpha}$ et latus $AB = a = \frac{-f\,fin.\,\alpha}{fin.\,(\alpha+2\beta+\psi)}$, et propter summam angulorum $\alpha+2\beta+\psi = ang.\,fin.\frac{-f\,fin.\,\alpha}{a}$ sequitur fore angulum $\beta = \tfrac{1}{2}\,ang.\,fin.\frac{-f\,fin.\,\alpha-}{a}\tfrac{1}{2}\alpha - \tfrac{1}{2}\psi$, et angulum $A = \psi = ang.\,fin.\frac{-f\,fin.\,\alpha}{a} - \alpha - 2\beta$ denique pro angulo ACB, $tang.\,\alpha = \frac{-a\,fin.\,(2\beta+\psi)}{f+a\,cos.\,(2\beta+\psi)}$

Sed

Sed ex aequatione posteriore, quae locum habet in casu evanescentis anguli ACD, nihil sequitur nisi solita analogia trigonometrica in triangulo ACB. $a : \text{sin.}\,\alpha = f : \text{sin.}\,(\alpha + \psi)$.

Scholion VI.

§. 117. Aequatio generalis hujus problematis pro trapezio directo non recipit varietates, nisi quae ex ambiguitate signi radicalis et cosinus oriuntur. Quod si trapezium ita sit partialiter inversum ut totum triangulum ADC, cadat intra triangulum ABC; habetur haec aequatio: $af\,\text{sin.}\,\beta = fc\,\text{sin.}\,\alpha\,\text{cos.}\,(\psi + \alpha - \beta) + c\,\text{sin.}\,(\psi + \alpha - \beta)\,\surd\,(a^2 - f^2\,\text{sin.}\,\alpha^2)$ Si trapezium sit totaliter inversum cadente latere AD intra latus AB, sed latere CD extra latus CB, eadem aequatio prodit. Sin autem latus AD cadat extra latus CB, latus vero CD intra latus CB, habetur haec aequatio: $af\,\text{sin.}\,\beta = fc\,\text{sin.}\,\alpha\,\text{cos.}\,(\alpha - \beta - \psi) + c\,\text{sin.}\,(\alpha - \beta - \psi)\,\surd\,(a^2 - f^2\,\text{sin.}\,\alpha^2)$. Quod si trapezium ita sit partialiter inversum ut totum triangulum ABC cadat intra triangulum ADC, eadem habetur aequatio; plures variationes aequatio non recipit, praeter eas, quae ex ambiguitate signi radicalis et cosinus oriuntur, quare haec est aequatio generalissima: $af\,\text{sin.}\,\beta = c\,\text{sin.}\,(\alpha \pm \psi \pm \beta)\,\surd\,(a^2 - f^2\,\text{sin.}\,\alpha^2) + fc\,\text{sin.}\,\alpha\,\text{cos.}\,(\alpha \pm \psi \pm \beta)$. In his autem aequationibus posui triangulum ADC, cadere ad sinistram diagonalis, quod si vero ponatur triangulum ABC cadere ad dextram diagonalis, similes respectu α et ψ variationes obtinebunt, quae igitur prodit, si in data hac aequatione generali α et β, loca sua permutent.

Problema IX.

Fig. IX. 1. §. 118. In figura quadrilatera propoſita $ABCD$ inter haec ſex; latus $BC=b$, $CD=d$, diagonalem $AC=f$, atque tres angulos $A=\psi$, $ACB=\alpha$, $ACD=\beta$, aequationem invenire.

Solutio.

1) Ex angulo B demiſſa in diagonalem perpendicularis Bb erit $=b\,\mathit{fin}.\,\alpha$, et ſegmentum $Cb=b\cos.\alpha$, et hinc ſegmentum reſiduum $Ab = f - b\cos.\alpha$, conſequenter latus $AB = \surd(f^2+b^2-2bf\cos.\alpha)$.

2) Hinc in triangulo ABb formo hanc analogiam: $1 : AB = \mathit{fin}.\,BAB : Bb$, h. e. $1 : \surd(f^2+b^2-2ab\cos.\alpha) = \mathit{fin}.\,BAb : b\,\mathit{fin}.\,\alpha$. Fit igitur: $\mathit{fin}.\,BAb = \frac{b\,\mathit{fin}.\,\alpha}{\surd(f^2+b^2-2ab\cos.\alpha}$ et $\cos.BAb$ ſive $\mathit{fin}.\,ABb = \frac{f-b\cos.\alpha)}{\surd(f^2+b^2-2ab\cos.\alpha)}$

3) Ex his et ex ſinu coſinuque anguli A, formo ſinum et coſinum anguli CAD, et obtineo $\mathit{fin}.\,CAD = \mathit{fin}.\,(\psi - CAB) = \frac{f\,\mathit{fin}.\,\psi - b\cos.\alpha\,\mathit{fin}.\,\psi - b\,\mathit{fin}.\,\alpha\cos.\psi}{\surd(f^2+b^2-2fb\cos.\alpha)}$ $= \frac{f\,\mathit{fin}.\,\psi - b\,\mathit{fin}.\,(\alpha+\psi)}{\surd(f^2+b^2-2bf\cos.\alpha}$ Atque $\cos.(\psi - CAB) = \frac{f\cos.\psi - b\cos.\alpha\cos.\psi + b\,\mathit{fin}.\,\alpha\,\mathit{fin}.\,\psi}{\surd(f^2+b^2-2fb\cos.\alpha)}$ $= \frac{f\cos.\psi - b\cos.(\alpha+\psi)}{\surd(f^2+b^2-2fb\cos.\alpha)} = \cos.CAd.$

4) Ex

4) Ex his et ex ſinu coſinuque anguli ACD, formo ſinum anguli D, et conſequor

$$fin.\,D = \frac{f\,fin.\,\beta\,cos.\,\psi - b\,fin.\,\beta\,cos.\,(\alpha+\beta) +}{\sqrt{(f^2+b^2-2fb\,cos.\,\alpha)}}$$

$$\frac{+ f\,cos.\,\beta\,fin.\,\psi - b\,cos.\,\beta\,fin.\,(\alpha+\psi)}{\sqrt{(f^2+b^2-2fb\,cos.\,\alpha)}}$$

$$= \frac{f\,fin.\,(\beta+\psi) - b\,fin.\,(\alpha+\beta+\psi)}{\sqrt{(f^2+b^2-2fb\,cos.\,\alpha)}}.$$ Hinc denique

5) in triangulo dextro pervenio ad hanc analogiam: $AC : fin. D = CD : fin. CAD$. h. e.

$$f : \frac{f\,fin.\,(\beta+\beta) - b\,fin.\,(\alpha+\beta+\psi)}{\sqrt{(f^2+b^2-2fb\,cos.\,\alpha)}} = d : \frac{f\,fin.\,\psi - b\,fin.\,(\alpha+\psi)}{\sqrt{(f^2+b^2-2fb\,cos.\,\alpha)}},$$ vel etiam quia per leges rationum denominator conſequentium excedit, habetur haec ſimplicior analogia: $f : f\,fin.\,(\beta+\psi) - b\,fin.\,(\alpha+\beta+\psi) = d : f\,fin.\,\psi\;b\,fin.\,(\alpha+\psi)$ adeoque $f d\,fin.\,(\beta+\psi) - b d\,fin.\,(\alpha+\beta+\psi) = f^2 fin.\,\psi - f b\,fin.\,(\alpha+\psi)$.

Scholion I.

§. 119. Quod ſi incipiendo in dextro triangulo formaſſem valores linearum Dd, Cd, et hinc linearum AD, Ad, et ex his ſinum et coſinum anguli CAB, et denique ſinum anguli B, perveniſſem ad hanc aequationem priori ſimilem et prorſus eandem terminis transpoſitis $f\,fin.\,\psi - f d\,fin.\,(\beta+\psi) = f b\,fin.\,(\alpha+\psi) - b d\,fin.\,(\alpha+\beta+\psi)$; poteſt vero ad plures alias aequationes perveniri, ſed haec eſt omnium ſimpliciſſima ad quam pervenire datur.

Coroll. I.

§. 120. Ex inventa aequatione obtinetur latus $CD = d = \frac{f^2 \text{fin.}\, \psi - f b\ \text{fin.}\, (\alpha + \psi)}{f \text{fin.}\, (\beta + \psi) - b\, \text{fin.}\, (\alpha + \beta + \psi)} = f \frac{(f\ \text{fin.}\, \psi - b\ \text{fin.}\, (\alpha + \psi)}{f\ \text{fin.}\, (\beta + \psi) - b\ \text{fin.}\, (\alpha + \beta + \psi)}$ et latus $BC = b = \frac{f d\ \text{fin.}\, (\beta + \psi) - f^2\ \text{fin.}\, \psi}{d\, \text{fin.}\, (\alpha + \beta + \psi) - f\, \text{fin.}\, (\alpha + \psi),} = f \frac{(d\ \text{fin.}\, (\beta + \psi) - f\ \text{fin.}\, \psi)}{d\ \text{fin.}\, (\alpha + \beta + \psi) - f\ \text{fin.}\, (\alpha + \psi)}$ sed pro diagonali habetur haec aequatio:

$$f = \frac{b\, \text{fin.} (\alpha+\psi) + d\, \text{fin.} (\beta+\psi) \pm \surd ((b\, \text{fin.} (\alpha+\psi) + d\, \text{fin.} (\alpha+\psi))^2 - 4 b d\ \text{fin.}\, \psi\ \text{fin.}\, (\alpha + \beta + \psi)}{2\ \text{fin.}\, \psi}$$

Coroll. II.

§. 121. Pro angulo A habetur haec aequatio:

$$tang. \psi = \frac{b d\ \text{fin.}\, (\alpha + \beta) - f b\, \text{fin.}\, \alpha - f d\ \text{fin.}\, \beta}{f d \cos. \beta + f b\ \cos. \alpha - b d\ \cos. (\alpha + \beta) - f^2}$$

et consequenter

$$\psi = ang. tang. \left(\frac{b d\, \text{fin.}\, (\alpha+\beta) - f b\ \text{fin.}\, \alpha - f d\, \text{fin.}\, \beta}{f d\, \cos. \beta + f b\, \cos. \alpha - b d\, \cos. (\alpha+\beta) - f^2} \right)$$

Explicata aequatione et posito brevitatis gratia $\text{fin.}\, (\beta + \psi) = x$, et $\cos. (\beta + \psi) = \surd (1 - x)$, $\text{fin.}\, \alpha = m$, $\cos.\, \alpha = n$, $f^2 \text{fin.}\, \psi - f b\ \text{fin.}\, (\alpha + \psi) = A$; erit factis substitutionibus aequatio talis: $(f - b n)\, d x - A = b d m \surd (1 - A^2)$, ex qua operatione absoluta obtinetur haecce aequatio: $x = A$

$$x = A\frac{(f-bn) \pm bm\sqrt{((f^2+b^2-2fbn)d^2-A^2)}}{d(f^2+b^2-2fbn)} = fin.(\beta+\psi)$$

et hinc ipfe angulus

$$\beta = ang.\,fin.\frac{(A(f-bn) \pm bm\sqrt{((f^2+b^2-2bfn)d^2-A^2)})-\psi}{d(f^2+b^2-2fbn)}$$

Coroll. III.

§. 122. Similiter pofito $fin.(\alpha+\psi) = x$, $cos.(\alpha+\psi) = \sqrt{(1-x^2)}\,fin.\,\beta = m$, $cos.\,\beta = n$, $f^2\,fin.\,\psi - -fd\,fin.\,(\beta+\psi) = A$ pervenio ad hanc aequationem fimilem priori:

$$x = \frac{A(f-dn) \pm dm\sqrt{(f^2+d^2-2fdn)b^2-A^2)}}{b(f^2+d^2-2fdn)} = fin.(\alpha+\psi)$$

et hinc ipfe angulus erit:

$$\alpha = ang.\,fin.\frac{(A(f-dn) \pm dm\sqrt{((f^2+d^2-2fdn)b^2-A)^2)})-\psi}{b(f^2+d^2fdn}$$

Scholion II.

§. 123. Ita fex illi cafus qui in aequatione implicantur funt evoluti, horum quinque utrique utrique methodo et trigonometricae et tetragonometricae fubfunt, quos ideo evolvi, quia fine labore ex aequatione eruuntur, fed facilius et brevius trigonometrice, quam tetragonometrice expediuntur. Solus cafus quo diagonalis quaeritur eft Tetragonometriae proprius, neque enim trigonometrice refolvi poteft, fuppeditat vero hic cafus problema in Geometria practica novum utile fimul pulcrum et egregium.

Coroll. IV.

§. 124. Si anguli diagonaliter oppositi ponantur simul sumti aequales duobus rectis, ut quadrilaterum sit circulo inscriptibile, prodibit pro diagonali haec aequatio ab omni irrationalitate libera: $f = \frac{b\,\text{fin.}\,(\alpha+\psi) + d\,\text{fin.}\,(\beta+\psi)}{\text{fin.}\,\psi}$.

Sed praeter hunc casum datur etiam alter, quo aequatio ab omni irrationalitate liberatur, si nimirum sit $(b\,\text{fin.}\,(\alpha+\psi) + d\,\text{fin.}\,(\beta+\psi))^2 = 4bd\,\text{fin.}\,\psi\,\text{fin.}\,(\alpha+\beta+\psi)$ ut aequatio fiat $f = \frac{b\,\text{fin.}\,(\alpha+\psi) + d\,\text{fin.}\,(\beta+\psi)}{2\,\text{fin.}\,\psi}$.

Coroll. V.

§. 125. Quod si anguli diagonaliter oppositi simul sumti uni recto aequentur, eliminato angulo ψ, aequatio inventa mutatur in hanc pro diagonali:

$$f = \frac{b\cos\beta + d\cos.\alpha \pm \sqrt{((b\cos.\beta + d\cos.\alpha)^2 - 4bd\cos.(\alpha+\beta))}}{2\cos.(\alpha+\beta)}.$$

Si anguli A et ABC sint simul sumti uni recto aequales, prodit eliminato ψ pro diagonali haec aequatio:

$$f = \frac{b + d\cos.(\beta-\alpha) \pm \sqrt{((b + d\cos.(\beta-\alpha))^2 - 4bd\cos.\alpha\cos.\beta}}{2\cos.\alpha}.$$

Si vero anguli A et ACD sint simul uni recto aequales, prodit haec aequatio:

$f =$

$$f=\frac{b\cos.(\alpha-\beta)+d\pm\surd((b\cos.(\alpha-\beta)+d^2)-4bd\cos.\alpha\cos.\beta}{2\cos.\beta}$$

Si ergo fit $\alpha=\beta$, aequationes mutantur in has: $f=\frac{b+d\pm\surd(b+d)^2-4bd\cos.\alpha^2)}{2\cos\alpha}$

vel etiam: $f=\frac{b+d\pm\surd((b+d)^2-4bd\cos.\beta^2)}{2\cos.\beta}$

quae in has immutari possunt:

$$f=\frac{b+d\pm\surd(b^2+d^2-2bd\cos.2\alpha)}{2\cos.\alpha}=\frac{b+d\pm\surd(b^2+d^2-2bd\cos.2\beta)}{2\cos.\beta}$$

Coroll. VI.

§. 126. Si ponatur angulus A aequalis duobus rectis, lateribus AB, AD, in diagonalem BD incidentibus, et trapezio in triangulum abeunte, aequatio primo inventa in hanc mutatur: $-fd\,\text{sin}.\beta+bd\,\text{sin}.(\alpha+\beta)=fb\,\text{sin}.\alpha$, ex qua elicitur, $f=bd\,\text{sin}.(\alpha+\beta):(b\,\text{sin}.\alpha+d\,\text{sin}.\beta)$, qui casus ex aequatione generali pro diagonali elici nequit ob $\text{sin}.\psi=\text{sin}.\pi=o$, resolvitur autem aequatio in hanc analogiam: $b\,\text{sin}.\alpha+d\,\text{sin}.\beta : b\,\text{sin}.(\alpha+\beta)=d:f$, quae hoc suppeditat Theorema.

Si in Triangulo BCD latus DC producatur, Fig. IX. 2. in productum ab angulo B demittatur normalis Bb, et in ductam ab angulo C intra triangulum quomodocunque rectam CA demittantur ab angulis B et D normales, erit summa perpendiculorum

lorum Dd, et Bc ad perpendiculum Bb uti latus DC ad rectam AC. Si ergo anguli α et β simul sumti sint uni recto aequales, sive angulus C rectus, et recta CA normalis in latus BD, prodit nota ex elementis Geometriae proprietas trianguli rectanguli: hypothenusa est ad crus unum, ut crus alterum ad perpendiculum.

Coroll. VII.

Fig. IX. 1. §. 127. Si angulus C sit aequalis duobus rectis, lateribus BC, DC in diagonalem BD cidentibus, et trapezio in triangulum BAD abeunte, sit aequatio pro recta AC, quae diagonali in quadrilatero jam respondet, hujusmodi:

$$f = \frac{b\,\sin.(\alpha+\psi) + d\,\sin.(\beta+\psi) \pm \surd(b\,\sin.(\alpha+\psi) + d\,\sin.(\beta+\psi))^2 + 4\,b\,d\,\sin.\psi^2)}{2\,\sin.\psi}$$

quae β eliminato mutatur in hanc:

$$f = \frac{b\,\sin.(\alpha+\psi) \pm d\,\sin.(\psi-\alpha) \pm \surd(b\,\sin.(\alpha+\psi) \pm d\,\sin.(\psi-\alpha))^2 + 4\,b\,d\,\sin.\psi^2)}{2\,\sin.\psi}$$

sed α eliminato habetur:

$$f = \frac{\pm b\,\sin.(\psi-\beta) + d\,\sin.(\beta+\psi) \pm \surd((d\,\sin.(\beta+\psi) \pm b\,\sin.(\psi-\beta))^2 + 4\,b\,d\,\sin.\psi^2)}{2\,\sin.\psi}$$

quae aequationes novam quandam triangulorum proprietatem exprimunt et problema circa triangula prorsus novum resolvunt, quod ita habet:

Si

Si ab angulo trianguli dato, ducatur ad latus oppositum recta, denturque segmenta lateris hujus cum angulis adjacentibus; invenire rectam in latus oppositum ductam AE, et resolvere triangulum totale ABD et partialia ABE, AED.

Coroll. VIII.

§. 128. Si angulus A ponatur aequalis tribus rectis, lateribus AB, AD supra diagonalem BD cadentibus, et prodeunte sic trapezio sive ex parte sive totaliter inverso, aequatio pro diagonali in hanc mutatur: $f = \frac{b \cos. \alpha + d \cos. \beta \pm}{2}$ $\frac{\surd(b \cos. \alpha + d \cos. \beta)^2 - 4\, b d \cos. (\alpha + \beta))}{2}$. Si vero angulus C ponatur aequalis tribus rectis, lateribus CB, CD infra diagonalem BD cadentibus, et trapezio ex parte vel totaliter inverso iterum prodeunte, habetur pro diagonali haec aequatio: $f = \frac{b \sin. (\alpha + \psi) + d \sin. (\beta + \psi)}{2 \sin. \psi}$ $\frac{\pm \surd((b \sin. (\alpha + \psi) + d \sin. (\beta + \psi) + 2\, b d \sin. 2\psi)}{2 \sin. \psi}$ sed β eliminato abit in hanc sequentem: $f = \frac{b \sin. (\alpha + \psi) - d \cos (\psi - \alpha) \pm \surd\, (b \sin. (\alpha + \psi) -}{2 \sin. \psi}$ $\frac{- d \cos. (\psi - a))^2 + 2\, b d \sin. 2\psi}{2 \sin. \psi}$ Sed α eliminato abit in hanc: $f = \frac{d \sin. (\beta + \psi) - b \cos. (\psi - \beta) \pm}{2 \sin. \psi}$ $\frac{\surd((d \sin. (\beta + \psi) - b \cos. (\psi - \beta))^2 + 2\, b d \sin. 2\psi)}{2 \sin. \psi}$ Quod

Quod ſi contingat angulum ACB vel ACD fieri aequalem angulo combinationis A, habetur in caſu priori haec aequatio:

$$f = \frac{b\,\text{ſin.}\,2\alpha - d \pm \surd(b^2\,\text{ſin.}\,2\alpha^2 + d^2)}{2\,\text{ſin.}\,\alpha},$$ et in

poſteriore $f = \dfrac{d\,\text{ſin.}\,2\beta - b \pm \surd(d^2\,\text{ſin.}\,2\beta^2 + b^2)}{2\,\text{ſin.}\,\beta.}$

Coroll. IX.

§. 129. Si angulus $ACB = \alpha$ ponatur negativus, ita ut ad dextram diagonalis cadat, vel etiam angulus $ACD = \beta$ negativus, ut ad ſiniſtram diagonalis cadat, uno triangulo intra alterum cadente; erit in caſu priore aequatio pro trapezio inverſo ad diagonalem hujusmodi:

$$f = \frac{\text{ſin}\,(\psi - \alpha) + d\,\text{ſin.}\,(\beta + \psi) \pm \surd(b\,\text{ſin.}\,(\psi - \alpha) +}{2\,\text{ſin.}\,\psi}$$

$$\frac{d\,\text{ſin.}\,(\beta + \psi))^2 - 4bd\,\text{ſin.}\,\psi\,\text{ſin.}\,(\beta + \psi) - \alpha))}{2\,\text{ſin.}\,\psi}$$

ſed in caſu poſteriore habetur:

$f = b\,\text{ſin.}\,(\alpha + \psi) + d\,\text{ſin.}\,(\psi - \beta) \pm \surd(b\,\text{ſin.}\,(\alpha + \psi)$ $+ d\,\text{ſin.}\,(\psi - \beta))^2 - 4bd\,\text{ſin.}\,\psi\,\text{ſin.}\,(\alpha + \psi - \beta))$.

Quod ſi ergo angulus ACB ſit major angulo ACD, vel etiam hic illo major, erit in utroque caſu trapezium totaliter inverſum, nam in priore caſu latus CB cadet extra latus CD, in poſteriore caſu latus CD extra latus CB, et conſequenter utraque diagonalis cadet extra figuram. Sin autem anguli illi ſunt aequales erit haec aequatio:

$f =$

$$f = \frac{b\,fin.(\psi-\alpha)+d\,fin.(\beta+\psi) \pm d\,fin.(\beta+\psi) \pm}{2\,fin.\,\psi}$$
$$\frac{\surd\big((b\,fin.(\psi-\alpha)+d\,fin.(\beta+\psi))^2-2bd(1-cos.\,2\psi)\big)}{2\,fin.\,\psi}$$

vel etiam:

$$f = \frac{b\,fin.(\beta+\psi)+d\,fin.(\psi-\beta) \pm \surd\big((b\,fin.(\beta+\psi)}{2\,fin.\,\psi}$$
$$\frac{+d\,fin.(\psi-\beta))^2-2\,b\,d\,(1-cos.\,2\psi)\big).}{2\,fin.\,\psi}$$

Scholion III.

§. 130. Rursus methodo tetragonometrica solvitur hac ratione problema novum ad triangula spectans, quod vix ulla alia methodo resolvitur; nam in triangulo ABD non dantur nisi duo scilicet angulus $A=\psi$, et latus ipsi oppositum BD, quod est differentia laterum BC, CD; nec in triangulo ADC datur nisi latus DC et angulus adjacens $ACD=\alpha=\beta$, tamen utrumque triangulum hac methodo solvitur, cum AC, quae diagonalem repraesentat, inveniatur. Fig. IX. 2.

Coroll. X.

§. 131. Si anguli diagonaliter oppositi ponantur simul tribus rectis aequales, ut caeteri diagonaliter oppositi B et D sint uni recto aequales, cadente vel A supra vel etiam C infra diagonalem BD, et trapezio inverso prodeunte, aequatio pro diagonali in hanc mutatur:

$$f = \frac{b\,fin.(\alpha+\psi)+d\,fin.(\beta+\psi) \pm \surd\big(b\,fin.(\alpha+\psi)}{2\,fin.\,\psi}$$
$$\frac{+d\,fin.(\beta+\psi))^2+4\,b\,d\,fin.\,\psi\big)}{2\,fin.\,\psi}$$ vel etiam haec

$f=$

$f = \frac{b\, cos.\, \beta + d\, cos.\, \alpha \pm \surd (b\, cos.\, \beta + d\, cos.\, \alpha)^2}{2\, cos.\, (\alpha + \beta)}$

$\frac{-- 4\, b\, d\, cos.\, (\alpha + \beta))}{2\, cos.\, (\alpha + \beta)}$. Per hanc poſitionem fit latus $CD = d = f \frac{(b\, cos.\, \beta - f\, cos.\, (\alpha + \beta))}{b - f\, cos.\, \alpha}$, et latus $B\,C == b == \frac{(f\, cos.\, (\alpha + \beta) -- d\, cos.\, \alpha) f}{f\, cos.\, \beta - d}$.

Quod ſi angulus A ponatur rectus, habetur pro diagonali:

$f = \frac{b\, cos.\, \alpha + d\, cos.\, \beta \pm \surd ((b\, cos.\, \alpha + d\, cos.\, \beta)^2}{2}$

$\frac{-- 4\, b\, d\, cos.\, (\alpha + \beta)}{2}$. Per hanc poſitionem fit latus $C\,D == d == \frac{f\,(f -- b\, cos.\, \alpha)}{f\, cos.\, \beta - b\, cos.\, (\alpha + \beta)}$ et latus $BC == \frac{f\,(d\, cos.\, \beta -- f)}{d\, cos.\, (\alpha + \beta) - f\, cos.\, \alpha} = b$. In aequatione pro angulo ACD, erit, $x == cos.\, \beta$, et $A == f^2 -- f\,b\, cos.\, \alpha$, quoad caetera aequationis quadraticae forma externa manet invariata. Similiter in aequatione pro angulo $A\,C\,B$, erit $x == cos.\, \alpha$, et $A == f^2 + f\, d\, cos.\, \beta$, forma vero externa manet invariata. Si angulus C ponatur aequalis recto, aequatio pro diagonali in hanc abit, α eliminato:

$\frac{b\, cos.\, (\psi - \beta) + d\, ſin.\, (\beta + \psi) \pm \surd ((b\, cos.\, (\psi - \beta)}{2\, ſin.\, \psi}$

$\frac{+ d\, ſin.\, (\beta + \psi)) -- 2\, b\, d\, ſin.\, 2\, \psi)}{2\, ſin.\, \psi}$ vel etiam, angulo β eliminato, in hanc alteram:

$f ==$

$$f = \frac{b \sin.(\alpha+\psi) + d \cos.(\psi-\alpha) \pm \surd((b \sin.(\alpha+\psi) + d \cos.(\psi-\alpha))^2 - 2bd \sin. 2\psi)}{2 \sin. \psi}.$$

In hoc caſu fit

latus $CD = d = f \frac{(f \sin. \psi - b \sin.(\alpha+\psi))}{f \sin.(\beta+\psi) - b \cos. \psi}$

vel etiam: $d = f \frac{(f \sin. \psi -- b \cos.(\psi-\beta))}{f \sin.(\beta+\psi) - b \cos. \psi} = f \frac{(f \sin. \psi - b \sin.(\alpha+\psi))}{f \cos.(\psi-\alpha) - b \cos. \psi}$. In hoc caſu fit

$\mathit{tang.}\, \psi = \frac{b(d - f \sin. \alpha) - f d \cos. \alpha}{f(d \sin. \alpha + b \cos. \alpha - f)}$, β eliminato, ſed α eliminato habetur:

$$\mathit{tang.}\, \psi = \frac{b(d -- f \cos. \beta) -- f d \sin. \beta}{f(d \cos. \beta + b \sin. \beta - f)}.$$

Si anguli A et ACB ponantur ſimul ſumti aequales duobus rectis, habetur pro diagonali haec expreſſio:

$$f = \frac{d \sin.(\beta+\psi) \pm \surd(d^2 \sin.(\beta+\psi)^2 + 4bd \sin. \beta \sin. \psi}{2 \sin. \psi},$$

α eliminato, ψ autem eliminato, habetur haec: $f = \frac{\mp d \sin.(\beta-\alpha) \pm \surd(d^2 \sin.(\beta-\alpha)^2 + 4bd \sin. \beta \sin. \alpha)}{2 \sin. \alpha}$. Si autem anguli A et ACD ponantur ſimul ſumti aequales duobus rectis, habetur pro diagonali haec aequatio:

$f = \frac{b\, \text{fin.}\, (\alpha+\psi) \pm \sqrt{}(b^2\, \text{fin.}\, (\alpha+\psi) + 4bd\, \text{fin.}\, \alpha\, \text{fin.}\, \psi)}{2\, \text{fin.}\, \psi}$, β eliminato, verum ψ eliminato, habetur haec:

$f = \frac{\mp b\, \text{fin.}\, (\alpha - \beta) \pm \sqrt{}(b^2\, \text{fin.}\, (\alpha - \beta)^2 + 4bd\, \text{fin.}\, \alpha\, \text{fin.}\, \beta}{2\, \text{fin.}\, \beta}$. Quare fi in utravis aequatione, ψ eliminato, ponatur $\alpha = \beta$ fit $f = \sqrt{}bd$, unde obtinetur hoc

Theorema.

Si in trapezio anguli ACB et ACD fint aequales inter fe, et fumma unius et anguli combinationis duobus rectis aequales, erit diagonalis AC media proportionalis inter latera, quibus anguli ifti continentur. Per pofitionem priorem fit latus $d = f\, \text{fin.}\, \psi : (c\, \text{fin.}\, \beta + f\, \text{fin.}\, (\beta+\psi)$, et latus $b = -f\, (d\, \text{fin.}\, (\beta+\psi) - f\, \text{fin.}\, \psi) : d\, \text{fin.}\, \beta$; verum ψ eliminato fit $d = f^2\, \text{fin.}\, \alpha : (b\, \text{fin.}\, \beta \mp f\, \text{fin.}\, (\beta - \alpha))$, atque $b = f(f\, \text{fin.}\, \alpha \mp d\, \text{fin.}\, (\beta - \alpha)) : d\, \text{fin.}\, \beta$. Per pofitionem pofteriorem fit latus

$d = f \frac{(\text{fin.}\, \psi - b\, \text{fin.}\, (\alpha + \psi))}{b\, \text{fin.}\, \alpha}$ et

$b = f^2\, \text{fin.}\, \psi : (d\, \text{fin.}\, \alpha + f\, \text{fin.}\, (\alpha+\psi))$, fed eliminato ψ fit $d = f \frac{(f\, \text{fin.}\, \beta \mp b\, \text{fin.}\, (\alpha - \beta))}{b\, \text{fin}\, \alpha}$

et $b = f^2\, \text{fin.}\, \beta : (d\, \text{fin.}\, \alpha \mp f\, \text{fin.}\, (\alpha - \beta))$

Schol.

Scholion IV.

§. 132. Aequatio hujus problematis pro trapezio directo valet in hypothesi figurae constructae, quando perpendiculum cadit intra figuram, sive quando angulus ACB est acutus, sed existente eodem obtuso et simul angulo A majore quam ACB, valet haec aequatio: $fd\,fin.(\beta+\psi)+bd\,fin.(\beta+\psi-\alpha)=f^2\,fin.\psi+fb\,fin.(\psi--\alpha)$: Sin autem angulus ACB excedat angulum A, valet haec aequatio: $fd\,fin.(\beta+\psi)--bd\,fin.(\alpha-\beta-\psi)=f^2\,fin.\psi--fb\,fin.(\alpha--\psi)$. Generalis igitur aequatio pro trapezio directo talis est: $fd\,fin.(\beta+\psi)\mp bd\,fin.(\pm\alpha\pm\beta\pm\psi)=f^2\,fin.\psi\mp fb\,fin.\begin{pmatrix}\psi\pm\alpha\\ \alpha--\psi\end{pmatrix}$

Si trapezium sit partialiter inversum ita ut totum triangulum ACD cadat intra triangulum ABC cadente perpendiculo Bb intra figuram, et existente angulo ACD majore quam angulo BAD; habetur haec aequatio: $fd\,fin.(\beta--\psi)--bd\,fin.(\beta-\alpha-\psi)=fb\,fin.(\alpha+\psi)--f^2\,fin.\psi$. Sed existente angulo ACD minore, quam est angulus BAD, caeteris manentibus iisdem, habetur haec aequatio: $--fd\,fin(\psi--\beta)+bd\,fin.(\psi+\alpha-\beta)==fb\,fin.(\alpha+\psi)-f\,fin.\psi$ Quod si perpendiculum cadat extra figuram, sitque angulus ACD major angulo BAD, et simul angulus ACB eodem major, habetur haec aequatio: $fd\,fin.(\beta--\psi)+bd\,fin.(\alpha+\beta--\psi)==f\,fin.(\alpha-\psi)--f\,fin.\psi$. Sin autem angulus ACB minor sit angulo BAD; terminus prior dextri membri etiam negativus erit; verum existente angulo BAD majore, quam est angulus ACD, et minore quam angulus ACB, habetur haec

 aequa-

aequatio: $-fd\,fin.(\psi-\beta)+bd\,fin.(\alpha+\beta-\psi)=f\,fin.(\alpha-\psi)-f^2\,fin.\psi$. Sin autem eodem minor ſit, haec obtinetur aequatio: $fd\,fin.(\psi-\beta)-bd\,fin.(\alpha+\beta-\psi)=fb\,fin.(\psi-\alpha)+f^2\,fin.\psi$. Si trapezium ſit totaliter inverſum, cadente perpendiculo intra figuram, latere AD extra latus AB, et latere BC extra latus CD, et exiſtente angulo ACB majore, quam angulo BAD, ſed minore, quam ſumma angulorum ACD et BAD, habetur haec aequatio: $fd\,fin.(\beta+\psi)-bd\,fin.(\beta+\psi-\alpha)=fb\,fin.(\alpha-\psi)+f^2\,fin.\psi$. Sin autem angulus BAD excedat angulum BAC erit: $fd\,fin.(\beta+\psi)-bd\,fin.(\beta+\psi-\alpha)=-fb\,fin.(\psi-\alpha)+f^2\,fin.\psi$. Denique ſi angulus ACB excedat ſummam angulorum ACD et BAD habetur haec aequatio: $fd\,fin.(\beta+\psi)+bd\,fin.(\alpha-\beta-\psi)=fb\,fin.(\alpha-\psi)+f^2\,fin.\psi$. Quod ſi perpendiculum cadat extra figuram, prodit haec aequatio: $fd\,fin.(\beta+\psi)+bd.\,fin.(\alpha+\beta+\psi)=fb\,fin.(\alpha+\psi)+f^2\,fin.\psi$. Si trapezium ita ſit partialiter inverſum, ut totum triangulum ABC cadat intra triangulum ADC, eaedem cum tribus ultimis aequationes prodibunt, neque plures jam allatis variationes locum habebunt. Igitur aequatio omnium generaliſſima repraeſentatur hoc modo: $\pm fd\,fin.\binom{\beta\pm\psi}{\psi-\beta}\mp bd\,fin.$

$$(\pm\beta\pm\alpha\pm\psi)=\pm(b\,fin.\binom{\alpha\pm\psi}{\psi-\alpha}\mp f^2\,fin.\psi.$$

Problema X.

Fig. X. 1. §. 133. In figura quadrilatera propoſita $ABCD$ inter haec ſex: latus $AB=a$, $CD=d$, diago-

diagonalem $A\,C = f$, et tres angulos $A = \psi$, $A\,C\,B = \alpha$, et denique $A\,C\,D = \beta$; aequationem invenire.

Solutio.

1) In triangulo finiſtro formo hanc analogiam: $AB\,(a) : \text{ſin.}\ ACB\ (\text{ſin.}\ \alpha) = AC\,(f)$;

$\text{ſin.}\ B = \frac{f\ \text{ſin.}\ \alpha}{a}$, conſequenter

$$\cos. B = \frac{\sqrt{(a^2 - f^2\ \text{ſin.}\ \alpha^2)}}{a}$$

2) Ex his et ex finu coſinuque anguli $A\,C\,B$, formo finum et cofinum anguli tertii in eodem triangulo, atque prodibit

$$\text{ſin.}\ C\,A\,B = \frac{\text{ſin.}\ \alpha \sqrt{(a^2 - f^2\ \text{ſin.}\ \alpha^2)} + f\ \text{ſin.}\ \alpha \cos.\ \alpha}{a}$$

nec non

$$\cos.\ C\,A\,B = \frac{-\cos.\ \alpha \sqrt{(a^2 - f^2\ \text{ſin.}\ \alpha^2)} + f\ \text{ſin.}\ \alpha^2}{a}.$$

3) Ex his et ex finu cofinuque anguli A formo finum et eofinum anguli $C\,A\,D$ in triangulo dextro, et ob $C\,A\,D = A - C\,A\,B$, habetur:

$$\text{ſin.}\,(\psi - CAB) = \frac{-\text{ſin.}\,\psi \cos.\,\alpha \sqrt{(a^2 - f^2\,\text{ſin.}\,\alpha^2)} + f\,\text{ſin.}\,\alpha^2\,\text{ſin.}\,\psi}{a}$$

$$\frac{\text{ſin.}\,\alpha \cos.\,\psi \sqrt{(a^2 - f^2\,\text{ſin.}\,\alpha^2)} - f\,\text{ſin.}\,\alpha \cos.\,\alpha \cos.\,\psi}{a}$$

$$= \frac{-\text{ſin.}\,(\alpha + \psi) \sqrt{(a^2 - f^2\,\text{ſin.}\,a^2)} - f\,\text{ſin.}\,\alpha \cos.\,(\alpha + \psi)}{a} = \text{ſin.}\,CAD$$

$$\text{et}\ \cos.\,(\psi - CAB) = \frac{-\cos.\,\alpha \cos.\,\psi \sqrt{(a^2 - f^2\,\text{ſin.}\,\alpha^2)} + f\,\text{ſin.}\,\alpha \cos.\,\psi}{a}$$

 $+\frac{\text{ſin.}\,\alpha}{}$

$$+\frac{\sin.\alpha\,\sin.\psi\sqrt{(a^2-f^2\sin.\alpha^2)}+f\sin.\alpha\cos.\alpha\sin.\psi}{a}$$

$$=\frac{-\cos.(\alpha+\psi)\sqrt{(a^2-f^2\sin.\alpha^2)}+f\sin.\alpha\sin(\alpha+\psi)}{a}=\cos.CAD.$$

4) Ex his et ex ſinu coſinuque anguli ACD, formo ſinum anguli D et obtineo:

$$\sin.D=\frac{-\sin.\beta\cos.(\alpha+\psi)\sqrt{(a^2-f^2\sin.\alpha^2)}+f\sin.\beta\sin.\alpha\sin.(\alpha+\psi)}{a}$$

$$-\frac{\cos.\beta\sin.(\alpha+\psi)\sqrt{(a^2-f^2\sin.\alpha^2)}-f\sin.\alpha\cos.\beta\cos.(\alpha+\psi)}{a}$$

$$=\frac{-\sin.(\alpha+\beta+\psi)\sqrt{(a^2-f^2\sin.\alpha^2)}-f\sin.\alpha\cos.(\alpha+\beta+\psi)}{a}$$

5) His factis in triangulo dextro pervenio ad hanc analogiam $AC:\sin.D=CD:\sin.CAD$.

$$f:\frac{-\sin.(\alpha+\beta+\psi)\sqrt{(a^2-f^2\sin.\alpha^2)}-f\sin.\alpha\cos.(\alpha+\beta+\psi))}{a}$$

$$=d:\frac{-\sin.(\alpha+\psi)\sqrt{(a^2-f^2\sin.\alpha^2)}-f\sin.\alpha\cos.(\alpha+\psi)}{a}$$

conſequenter habetur:

$$(f\sin.(\alpha+\psi)-d\sin.(\alpha+\beta+\psi))\sqrt{(a^2-f^2\sin.\alpha^2)}$$
$$=f\sin.\alpha(d\cos.(\alpha+\beta+\psi)-f\cos.(\alpha+\psi))\ \text{Q.E.I.}$$

Scholion I.

§. 134. Cum ſumma angulorum B et D ſit $=2\pi-\alpha-\beta-\psi$, erit $\sin.(B+D)=-\sin.(\alpha+\beta+\psi)$ et $\cos.(B+D)=\cos.(\alpha+\beta+\psi)$, et cum ſit $\sin.B=\frac{f\sin.\alpha}{a}$ et $\cos.B=\frac{\sqrt{(a^2-f\sin.\alpha^2)}}{a}$; erit

$$\sin.(B+D-B)=\sin.D=$$
$$=\frac{-f\sin.\alpha\cos.(\alpha+\beta+\psi)-\sin.(\alpha+\beta+\psi)\sqrt{(a^2-f^2\sin.\alpha^2)}}{a}$$

con-

consequenter ad eandem ac prius, paullo brevius, aequationem pervenio:

$$(f\,fin.(\alpha+\psi)-d\,fin.(\alpha+\beta+\psi))\sqrt{(a^2-f^2 fin.\alpha^2)} = f\,fin.\alpha\,(d\,cos.(\alpha+\beta+\psi)-f\,cos.(\alpha+\psi))$$

Coroll. I.

§. 135. Ex hac aequatione invenitur esse latus $AB = a = \frac{f\,fin.\alpha\sqrt{(f^2+d^2-2fd\,cos.\beta)}}{f\,fin.(\alpha+\psi)-d\,fin.(\alpha+\beta+\psi)}$ atque latus $D\ C = d =$

$$= \frac{f(fin.(\alpha+\psi)\sqrt{(a^2-f^2 fin.\alpha^2)}+f\,fin.\alpha\,cos.(\alpha+\psi))}{fin.(\alpha+\beta+\psi)\sqrt{(a^2-f^2 fin.\alpha^2)}+f\,fin.\alpha\,cos.(\alpha+\beta+\psi)}$$

Coroll. II.

§. 136. Pro tangente anguli A obtinendo: $fin.(\alpha+\psi)$ et $cos.(\alpha+\psi)$ separatis, et aequatione reducta obtinetur:

$$tang.(\alpha+\psi) = \frac{(d\,cos.\beta-f\,fin.\alpha)f+d\,fin.\beta\sqrt{(a^2-f^2 fin.\alpha^2)}}{fd\,fin.\beta+(f-d\,cos.\beta)\sqrt{(a^2-f^2 fin.\alpha^2)}}$$

et hinc ad angulum ipsum descenditur:

$$\psi = ang.\ tang.\ \frac{(d\,cos.\beta-f\,fin.\alpha)f+d\,fin.\beta\sqrt{(a^2-f^2 fin.\alpha^2)}}{fd\,fin.\beta+(f-d\,cos.\beta)\sqrt{(a^2-f^2 fin.\alpha^2)}}-\alpha$$

Coroll. III.

§. 137. Posito ad abbreviandum $fin.(\alpha+\psi)=p$, $cos.(\alpha+\psi)=n$, $fin.(\alpha+\beta+\psi)=m$, $cos.(\alpha+\beta+\psi)=n$, $fin.\alpha=h$, et factis substitutionibus, erit aequatio: $(fp-dm)\sqrt{(a^2-f^2h^2)}=fh\,(dn-fg)$, quae sumtis quadratis, et operatione ad finem per-

 ducta,

ducta, abit in hanc aequationem quarti gradus:

$$f^4 - 2f^3 d\,cos.\,\beta + f^2(d^2 - \frac{a^2p^2}{h^2}) + \frac{2a^2fdmp}{h^2} - \frac{a^2d^2m}{h^2} = 0$$

Scholion II.

§. 138. Inter ſex illos caſus, qui in aequatione continentur, unus tantum eſt Tetragonometriae proprius, qui per Trigonometriam non ſolvitur, et in Geometria practica novum pulchrum et utile ſiſtit problema. caeteri quinque caſus utrique methodo et trigonometricae et tetragonometricae ſubſunt; ſed ita ut per Trigonometriam brevius et facilius ſolvantur, quorum caſuum tres evolvi, quia ex aequatione ſatis expedite ſequuntur. Caeteros duos angulorum partialium *ACB* et *ACD* omiſi, tum quod ſint inutiles, tum etiam quod operoſius evolvantur, et aequationibus non ſatis ſimplicibus et concinnis exprimantur.

Coroll. IV.

§. 139. Si anguli diagonaliter oppoſiti *A*, *C* ſint ſimul ſumti aequales duobus rectis, ut quadrilaterum ſit circulo inſcriptibile, aequatio mutatur in hanc quadraticam:

$f^2 - 2fd\,cos.\,\beta + d^2 - \frac{fin.\,\beta^2}{fin.\,\alpha} = 0$, ex qua obtinetur: $f = d\,cos.\,\beta \pm \frac{fin.\,\beta}{fin.\,\alpha}\sqrt{(a^2 - d^2 fin.\,\alpha^2)}$. Quod ſi angulus $ACB = \alpha$ ponatur nihilo aequalis, abeunte quadrilatero in triangulum *ACD*, aequatio pro diagonali in hanc abit:

f^2

$f^2 a^2$ *fin.* $\psi^2 - 2 f a^2 d$ *fin.* ψ *fin.* $(\beta + \psi) + a^2 d^2$ *fin.* $(\beta + \psi)^2 = o$, quae per a^2 divifa eft quadratum, cujus extracta radix eft f *fin.* $\psi - d$ *fin.* $(\beta + \psi) = o$ adeoque, f *fin.* $\psi == d$ *fin.* $(\beta + \psi)$ et *fin.* ψ: *fin.* $(\beta + \psi) = d : f$, analogia trigonometrica, quo ergo aequatio et operatio verificatur.

Coroll. V.

§. 140. Quod fi angulus A ponatur aequalis duobus rectis, incidente vertice A in diagonalem BD, et trapezio in triangulum BCD abeunte, aequatio pro diagonali in hanc mutatur:

$$f^4 \pm f^3 d \cos. \beta + f^2 (d^2 - a^2) + \frac{2 f a^2 d \, fin. (\alpha + \beta)}{fin. \, \alpha}$$

$$\frac{- a^2 d^2 fin. (\alpha + \beta)}{fin. \, \alpha^2} == o.$$

Scholion III.

§. 141. Haec aequatio novum folvit ad triangula fpectans problema hujusmodi:

Si in triangulo BCD detur latus CD, angulus C, et a vertice C in latus tertium ducta quomodocunque recta CE, detur etiam fegmentum BE, invenire rectam EC, et refolvere triangula partialia et totale. In triangulorum partialium neutro dantur nifi duo, nec in totali dantur nifi latus unum & angulus unus, & pars lateris alterius BE, quod igitur problema vix ulla alia methodo, quam tetragonometrica, refolvitur:

Coroll. VI.

§. 142. Quod fi angulus totalis C, five partialis fimul fumti α et β ponantur aequales duobus

bus rectis, lateribus CB, CD in diagonalem BD incidentibus, et trapezio in triangulum abeunte, prodit haec aequatio:

$$f^4 + 2f^3 d\, cos.\, \beta + f^2 \left(d^2 - \frac{a^2\, fin.\, (\alpha + \psi)^2}{fin.\, \alpha^2}\right) - \frac{2 f a^2 d\, fin.\, \psi\, fin.\, (\alpha + \psi)}{fin.\, \alpha^2} - \frac{a^2 d^2\, fin.\, \psi^2}{fin.\, \alpha^2} = 0.$$

Scholion IV.

§. 143. Haec aequatio quarti gradus ſolvit alterum novum ad triangula ſpectans problema tale: Si in triangulo ABD datur latus AB et angulus A, et a vertice A ducta recta AE, dentur anguli partiales apud E, et ſegmentum CD lateris BD; invenire rectam AE et reſolvere triangula; quod problema methodis vulgaribus non ſolvitur.

Coroll. VII.

§. 144. Si angulus A ponatur aequalis tribus rectis, cadente vertice A ſupra diagonalem BD, intra vel extra triangulum BCD, et prodeunte ſic trapezio ex parte vel totaliter inverſo, aequatio pro diagonali AC talis prodit:

$$f^4 - 2f^3 d\, cos.\, \beta + f^2 (d^2 - a^2\, cot.\, \alpha^2) + 2 f a^2 d\, cot.\, \alpha\, cos.\, (\alpha + \beta) - \frac{a^2 d^2\, cos.\, (\alpha + \beta)^2}{fin.\, \alpha^2} = 0.$$

Sed poſito angulo totali C, aequali tribus rectis, cadente C infra diagonalem BD, intra vel extra triangulum BAD, trapezio inverſo denuo prodeunte, pro diagonali AC habetur haec aequatio:

$$f^4 + 2f^3 d \,\text{fin.}\, \alpha + f^2 \frac{(d^2 - a^2 \,\text{fin.}\, (\alpha+\psi)^2) -}{\text{fin.}\, \alpha^2}$$

$$-2f^2 a d \cos.\, \psi \frac{\text{fin.}\, (\alpha+\psi)}{\text{fin.}\, \alpha^2} - \frac{a^2 d^2 \cos.\, \psi^2}{\text{fin.}\, \alpha^2} = 0.$$

Dum igitur anguli A et C crefcunt figillatim fupra duos rectos per totum tertium quadrantem ad tres rectos, trapezia funt ex parte inverfa; et fi angulus adhuc ulterius augeatur, trapezium inverfum continuat, donec latera inferiora cum fuperioribus tantum non coincidant.

Coroll. VIII.

§. 145. Si angulus $ACB = \alpha$ negativus ponatur, ut triangulum ABC cadat etiam ad dextram diagonalis AC, habetur aequatio pro diagonali AC extra trapezium cadente hujusmodi:

$$f^4 - 2f^3 d \cos.\, \beta + f^2 \frac{(d^2 - a^2 \,\text{fin.}\, (\psi-\alpha)^2)}{\text{fin.}\, \alpha^2} + 2fa^2 d$$

$$\frac{\text{fin.}\, (\beta - \alpha + \psi)\, \text{fin.}\, (\psi - \alpha)}{\text{fin.}\, \alpha^2} - \frac{a^2 d^2 \,\text{fin.}\, (\beta - \alpha + \psi)^2}{\text{fin.}\, \alpha} = 0;$$

atque hic eft cafus tertius, quo prodit trapezium inverfum. Quod fi denique ponatur angulus $ACD = \beta$ negativus, cadente triangulo ADC etiam ad finiftrum diagonalis, quarta vice prodit trapezium inverfum, pro cujus diagonali AC erit aequatio:

$$f^4 - 2f^3 d \cos.\, \beta + f^2 \frac{(d^2 - a^2 \,\text{fin.}\, (\alpha+\psi)^2)}{\text{fin.}\, \alpha^2}$$

$$+ 2fa^2 d \,\text{fin.}\, \frac{(\alpha - \beta + \psi)\, \text{fin.}\, (\alpha+\psi)}{\text{fin.}\, \alpha^2}$$

$$- a^2 d^2 \,\text{fin.}\, \frac{(\alpha - \beta + \psi)}{\text{fin.}\, \alpha^2}$$

Coroll.

Coroll. IX.

§. 146. Si anguli ACB et ACD ſint inaequales et ambo latera trianguli unius intra triangulum alterum cadunt, trapezium erit ex parte inverſum, ſed ſi unum intra unum, alterum extra alterum cadat, trapezium erit totaliter inverſum. Quod ſi vero anguli iſti ſint aequales, prodit pro diagonali haec aequatio:

$$f^4 - 2f^3 d \cos.\alpha + f^2 \frac{(d^2 - a^2 \sin.(\psi - \alpha)^2)}{\sin.\alpha^2} + \frac{2fda^2 \sin.\psi \sin.(\psi - \alpha)}{\sin.\alpha^2} - \frac{a^2 d^2 \sin.\psi^2}{\sin.\alpha^2} = 0.$$

Scholion V.

Fig. X. 2. §. 147. Siquidem latera CB, CD ſint inaequalia, haec aequatio quarti gradus novum ad triangula ſpectans ſolvit problema, hujusmodi. Si in triangulo totali ADC datur latus DC et angulus adjacens C et in triangulo partiali DAB, latus AB et angulus DAB, et in altero partiali ABC, latus AB et angulus oppoſitus C, invenire rectam AC, et reſolvere triangulum.

Coroll. X.

§. 148. Quod ſi angulus ACB ponatur aequalis angulo A, aequatio octavi Corollarii in hanc mutatur: $f^4 - 2f^3 d \cos.\beta + \frac{f^2 d^2 - a^2 d^2 \sin.\beta^2}{\sin.\alpha^2} = 0.$

Sin autem angulus ACD angulo A ponatur aequalis, aequatio poſterior in hanc abibit:

$f^4 -$

$$f^4 - 2f^3\, d \cos.\beta + f^2 \left(d^2 - \frac{a^2 \sin.(\alpha+\beta)^2)}{\sin.\alpha^2}\right.$$

$$+ \frac{2fa^2 d \sin.(\alpha+\beta) - a^2 d^2}{\sin.\alpha} = 0.$$

Coroll. XI.

§. 149. Si anguli diagonaliter oppofiti A, C ponantur fimul tribus rectis aequales, prodit pro diagonali haec aequatio:

$$f^4 - 2f^3\, d \cos.\beta + f^2 \frac{(d^2 - a^2 \cos.\beta^2)}{\sin.\alpha^2}$$

$$+ \frac{2fa^2 d \cos.\beta - a^2 d^2}{\sin.\alpha^2} = o,$$ et fi fumma angulorum A et ACB fit aequalis duobus rectis, aequatio pro diagonali mutatur in hanc:

$$f^4 - 2f^3 d \cos.\beta + f^2 d^2 - \frac{a^2 d^2 \sin.\beta^2}{\sin.\alpha^2} = o;$$ fi denique anguli A, ACD fint fimul fumti aequales duobus rectis, prodit haec aequatio:

$$f^4 - 2f^3\, d \cos.\beta + f^2 (d^2 - \frac{a^2 \sin.(\alpha - \beta^2))}{\sin.\alpha^2}$$

$$\pm 2f \frac{a^2 d \sin.(\alpha-\beta) - a^2 d^2}{\sin.\alpha} = 0.$$

Scholion VI.

§. 150. Si loco laterum AB, DC aequationem ingredientium latera BC, AD adhibeantur, ad hanc aequationem pervenio fimilem priori, et formaliter et fpecifice eandem:

$$(f \sin.(\beta+\psi) - b \sin.(\alpha+\beta+\psi)) \sqrt{(c^2 - f^2 \sin.\beta^2)}$$
$$= f \sin.\beta\, (b \cos.(\alpha+\beta+\psi) - f \cos.(\beta+\psi)).$$ Haec autem

autem ex priore prodit, si litteris α, d, a, respective substituantur litterae β, b, c, proinde quod supra §. 17. assertum erat, hic confirmatur, casus laterum dextri et sinistri, et laterum supremi et infimi non esse formaliter et specifice diversos. Caeterum casus, quo in uno triangulo aequationem ingreditur latus alterutrum et angulus diagonali oppositus in altero, Clariss. Lambert, si aequationis utilitatem respicias, satis recte praetermissus et rejectus videtur, cum inter sex casus, qui in aequatione continentur, nullus sit tetragonometriae proprius: tamen hi casus suppeditant aequationes tetragonometricas a caeteris omnibus formaliter & specie diversas, e quibus solutiones eliciuntur, quae trigonometricis parum aut nihil cedunt. Si aequationem ingrediatur latus sinistrum prodit haec aequatio: $f\,\mathit{sin.}\,\beta = -a\,\mathit{sin.}\,(\alpha+\beta+\phi+\psi)$; si vero latus supremum aequationem ingrediatur, haec prodit: $b\,\mathit{sin.}\,(\alpha+\beta+\phi+\psi) = f\,\mathit{sin.}\,(\beta+\phi+\psi)$.

Scholion VII.

§. 151. Aequatio hujus problematis pro trapezio directo valet in hypothesi figurae constructae, existente summa angulorum B et ACB recto majore; quod si vero eadem sit recto minor, ut cosinus summae eorundem affirmative sit accipiendus, et praeterea angulus A excedat angulum ACB, habetur haec aequatio: $(f\,\mathit{sin.}\,(\psi-\alpha) - d\,\mathit{sin.}\,(\beta+\psi-\alpha))\sqrt{(a^2 - f^2\,\mathit{sin.}\,\alpha^2)} = f\,\mathit{sin.}\,\alpha\,(f\,\mathit{cos.}\,(\psi-\alpha) - d\,\mathit{cos.}\,(\beta+\psi-\alpha^2))$. Sin autem angulus ACB excedat angulum A, habetur haec aequatio, siquidem idem minor

existat

existat summa angulorum A et $A\,C\,D$: $-(f\,sin.(\alpha-\psi)+d\,sin.(\beta+\psi-\alpha))\sqrt{(a^2-f^2\,sin.\,\alpha^2)}$ $=f\,sin.\,\alpha\,(f\,cos.\,(\alpha-\psi)-d\,cos.\,(\beta+\psi-\alpha))$. Si vero angulus $A\,C\,B$ excedat summam angulorum A et $A\,C\,D$, habetur haec: $(-f\,sin.(\alpha-\psi)+d\,sin.(\alpha-\beta-\psi))\sqrt{(a^2-f^2\,sin.\,\alpha^2)}$ $=f\,sin.\,\alpha(f\,cos.(\alpha-\psi)-d\,cos.(\alpha-\beta-\psi))$. Est igitur haec aequatio generalis pro omni trapezio directo:

$$(\pm f\,sin.\binom{\alpha\pm\psi}{\psi-\alpha}\mp d\,sin.(\pm\beta\pm\psi\pm\alpha))\sqrt{(a^2-f^2\,sin.\,\alpha^2)}$$

$$=f\,sin.\,\alpha(\mp f\,cos.\binom{\alpha\pm\psi}{\psi-\alpha}\pm d\,cos.(\pm\beta\pm\psi\pm\alpha))$$

Si trapezium sit partialiter inversum, ut totum triangulum $A\,D\,C$ cadat intra triangulum $A\,B\,C$, summa angulorum B et ACB excedente rectum, prodit haec aequatio: $(f\,sin.(\alpha+\psi)+d\,sin.(\alpha+\psi-\beta))\sqrt{(a^2-f^2\,sin.\,\alpha^2)}$ $=f\,sin.\,\alpha(-f\,cos.(\alpha+\psi)+d\,cos.(\alpha+\psi-\beta))$. Verum excedente recto summam angulorum B et ACB et simul angulo ACB angulum BAD, prodit haec: $(f\,sin.(\alpha-\psi)-d\,sin.(\alpha+\beta-\psi))\sqrt{(a^2-f^2\,sin.\,\alpha^2)}$ $=f\,sin.\,\alpha\,(-f\,cos.\,(\alpha-\psi)+d\,cos.\,(\alpha+\beta-\psi))$ Sed angulo $B\,A\,D$ excedente angulum $A\,C\,B$, habetur haec aequatio: $-(f\,sin.(\psi-\alpha)+d\,sin.(\alpha+\beta-\psi))\sqrt{(\alpha^2-f^2\,sin.\,\alpha^2)}$ $=f\,sin.\,\alpha\,(-f\,cos.\,(\psi-\alpha)+d\,cos.\,(\alpha+\beta-\psi))$. Verum excedente angulo $B\,A\,D$, summam angulorum $A\,C\,B$ et $A\,C\,D$, habetur haec: $-(f\,sin.\,(\psi-\alpha)-d\,sin.(\psi-\alpha-\beta))\sqrt{(a^2-f^2\,sin.\,\alpha^2)}$ $=f\,sin.\,\alpha\,(-f\,cos.\,(\psi-\alpha)+d\,cos.\,(\psi-\alpha-\beta))$ Quod si trapezium ita sit totaliter inversum ut latus $A\,D$ cadat intra latus $A\,B$, latus autem $C\,D$ extra latus $C\,B$, eaedem non aliae variationes occurrunt. Si trapezium ita sit totaliter inversum,

ſum, ut latus AD cadat extra latus AB, latus autem CB extra latus CD, exiſtente ſumma angulorum BAD et ACB majore, quam recto, et angulo BAD excedente angulum ACB, recurrit aequatio hujus ſcholii prima. Verum excedente angulo ACB, angulum BAD, ſumma tamen angulorum ACD et BAD ipſum ſuperante; prodit aequatio hujus ſcholii ſecunda. Sed excedente angulo ACB non modo angulum BAD, verum etiam ſummam angulorum BAD et ACD, habetur haec aequatio: $(f\,\text{ſin.}(\alpha-\psi)-d\,\text{ſin.}(\alpha-\beta-\psi))\sqrt{}(a^2-f^2\,\text{ſin.}\,\alpha^2)$ $=f\,\text{ſin.}\,\alpha(-f\cos.(\alpha-\psi)+d\cos.(\alpha-\beta-\psi))$. Caeterum exiſtente ſumma angulorum B et ACB recto minore, prodit aequatio in reſolutione problematis inventa. Generaliſſima igitur aequatio aliter exprimi nequit, quam generaliter expreſſa fuit pro trapezio directo.



CAPUT

CAPUT V.

Continens novem problemata tertiae claſſis particularis, ſub priore principali contentae.

Problema XI.

§. 152. In figura quadrilatera propoſita $ABCD$, inter haec ſex: latus $CD = d$, latus $BC = b$, diagonalem $AC = f$, angulum $A = \psi$, angulum $B = \lambda$, et denique angulum $ACD = \beta$; aequationem invenire. Fig. XI.

1) Cum in triangulo ſiniſtro ſit

$$\text{ſin.}\, CAB = \frac{b\, \text{ſin.}\, \lambda}{f} \text{ et } \cos.\, CAB = \frac{\sqrt{(f^2 - b^2 \text{ſin.}\, \lambda^2)}}{f}$$

ex his et ex ſinu coſinuque anguli A, formo ſinum et coſinum anguli CAD in triangulo dextro, et erit $\text{ſin.}\, (A - CAB) = \text{ſin.}\, (\psi - CAB) =$

$$= \frac{\text{ſin.}\, \psi \sqrt{(f^2 - b^2\, \text{ſin.}\, \lambda^2)} - b\, \text{ſin.}\, \lambda\, \cos.\, \psi}{f}$$

$= \text{ſin.}\, CAD$; et $\cos.\, (A - CAB) = \cos.\, (\psi - CAB) =$

$$= \frac{\cos.\, \psi \sqrt{(f^2 - b^2\, \text{ſin.}\, \lambda^2)} + b\, \text{ſin.}\, \psi\, \text{ſin.}\, \lambda}{f}$$

$= \cos.\, CAD$.

2) Ex his et ex ſinu coſinuque anguli ACD, formo ſinum anguli D, qui igitur erit:

$$= \frac{\text{ſin.}\, \beta \cos.\, \psi \sqrt{(f^2 - b^2\, \text{ſin.}\, \lambda^2)} + b\, \text{ſin.}\, \beta\, \text{ſin.}\, \psi\, \text{ſin.}\, \lambda}{f}$$

$$\frac{+ \cos.\, \beta\, \text{ſin.}\, \psi \sqrt{(f^2 - b^2\, \text{ſin.}\, d^2)} - b \cos.\, \beta \cos.\, \psi\, \text{ſin.}\, \lambda}{f}$$

 =

$$= \frac{\sin.(\beta+\psi)\sqrt{(f^2-b^2\sin.\lambda^2)}-b\cos.(\beta+\psi)\sin.\lambda}{f}$$

$= \sin. D.$

3) His factis in triangulo dextro pervenio ad hanc analogiam, $AC : \sin. D = CD : \sin. CAD$ h. e. symbolis substitutis:

$$f : \frac{\sin.(\beta+\psi)\sqrt{(f^2-b^2\sin.\lambda^2)}-b\sin.\lambda\cos.(\beta+\psi)}{f} =$$

$$d : \frac{\sin.\psi\sqrt{(f^2-b^2\sin.\lambda^2)}-b\sin.\lambda\cos.\psi}{f}$$

consequenter habetur haec aequatio:

$$(d\sin.(\beta+\psi)-f\sin.\psi)\sqrt{(f^2-b^2\sin.\lambda^2)} = b\sin.\lambda\,(d\cos.(\beta+\psi)-f\cos.\psi)$$ Q. E. I.

Coroll. I.

§. 153. Ex hac aequatione facillime obtinetur: latus $CD = d =$

$$= \frac{f(\sin.\psi\sqrt{(f^2-b^2\sin.\lambda^2)}-b\sin.\lambda\cos.\psi)}{\sin.(\beta+\psi)\sqrt{(f^2-b^2\sin.\lambda^2)}-b\sin.\lambda\cos.(\beta+\psi)},$$

nec difficulter invenitur latus BC, cum sit absoluta operatione $b = \frac{f(d\sin.(\beta+\psi)-f\sin.\psi)}{\sin.\lambda\sqrt{(f^2+d^2-2fd\cos.\beta)}}$

Scholion I.

§. 154. Cum $\sqrt{(f^2+d^2-2fd\cos.\beta)}$ sit valor lateris AD, existente angulo ACD acuto, sed obtuso existente sit $\sqrt{(f^2+d^2+2fd\cos.\beta)}$, signum autem negativum cosinus anguli β ortum sit e signo negativo $\sin.\psi$ et $\cos.\psi$, sequitur fieri posse, ut signo etiam affirmativo afficiantur, quod igitur in eo casu evenire debet, quo angu-

lus

lus ACD eſt obtuſus, conſequenter aequatio ita generalis repraeſentatur: $(d\,\mathit{ſin.}(\beta+\psi) \mp f\,\mathit{ſin.}\psi) \sqrt{(f^2-b^2\,\mathit{ſin.}\lambda^2)} = b\,\mathit{ſin.}\lambda\,(d\cos.(\beta+\psi) \mp f\cos.\psi)$ Sed ob ambiguitatem ſigni radicalis in ſiniſtro et coſinuum in dextro aequationis membro, ſequitur etiam accidere poſſe, ut aequatio repraeſentanda ſit hoc modo: $(f\,\mathit{ſin.}\,\psi - d\,\mathit{ſin.}\,(\beta+\psi)) \sqrt{(f^2-b^2\,\mathit{ſin.}\,\lambda^2)} = b\,\mathit{ſin.}\lambda\,(f\cos.\psi - d\cos.(\beta+\psi))$.

Coroll. II.

§. 155. Poſito ad abbreviandum $\mathit{ſin.}(\beta+\psi)=p$, $\cos.(\beta+\psi)=q$, $\mathit{ſin.}\psi=m$, $\cos.\psi=n$, $\mathit{ſin.}\,\lambda=h$, aequatio primo inventa mutatur, factis ſubſtitutionibus, in hanc abbreviatam: $(d\,p - f\,m) \sqrt{(f^2 - b^2\,h^2)} = b\,h\,(d\,q - f\,n)$, ex qua, ſumtis quadratis, concinnatione et reductione, obtinetur haec aequatio quarti gradus pro invenienda diagonali:

$$f^4 - \frac{2 f^3 d\,p}{m} + f^2\,\frac{(d^2\,p^2 - b^2\,h^2)}{m^2} + \frac{2\,f\,d\,b^2\,h^2\cos.\,\beta}{m^2} - \frac{b^2\,d^2\,h^2}{m^2} = 0.$$

Coroll. III.

§. 156. Sit ad abbreviandum quantitas ſub ſigno radicali $\sqrt{(f^2-b^2\,\mathit{ſin.}\lambda^2)} = M$, et pro obtinendo angulo combinationis habetur haec aequatio: $tang.\,\psi = \frac{M\,d\,\mathit{ſin.}\,\beta + b\,\mathit{ſin.}\,\lambda\,(f - d\cos.\,\beta)}{M\,(f - d\cos.\beta) - b\,d\,\mathit{ſin.}\,d\,\mathit{ſin.}\,\beta}$ et hinc ipſe angulus A

$$\psi = ang.\ tang.\frac{M\,d\,\mathit{ſin.}\beta + b\,\mathit{ſin.}\lambda\,(f - d\cos.\beta)}{M(f - d\cos.\beta) - b\,d\,\mathit{ſin.}\lambda\,\mathit{ſin.}\beta}.$$

Pro angulo B ex Coroll. I. ſtatim apparet eſſe:

$$\text{ſin. } \lambda = \frac{f(d\,\text{ſin.}(\beta+\psi) - f\,\text{ſin.}\,\psi)}{b\sqrt{(f^2+d^2 \mp 2fd\cos.\beta)}}.$$

Scholion II.

§. 157. Sex caſuum in aequatione contentorum nullus eſt Tetragonometriae proprius niſi unus, qui obtinet quando diagonalis quaeritur; caeteri quinque utrique methodo et trigonometricae et tetragonometricae pariter ſubſunt, ſed ita ut trigonometrice fere facilius ſolvantur; horum caſuum quatuor evolvi, cum ex aequatione ſatis expedite ſequantur, et per aequationes ſatis breves et concinnas exprimantur, omiſſo caſu, quo quaeritur angulus ACD, cum operoſius evolvatur, et per aequationem quadraticam quidem, ſed complicatam et haud ſatis concinnam exprimitur. Caeterum caſus quo diagonalis quaeritur, in Geometria practica ſuppeditat problema novum utile juxta ac ſatis pulchrum.

Coroll. IV.

§. 158. Si angulus A ſit rectus, aequatio pro diagonali, denominatore in unitatem abeunte, fit paullo ſimplicior: $f^4 - 2f^3 d\cos.\beta + f^2(d^2\cos.\beta^2 - b^2h^2) + 2fdb^2h^2\cos.\beta - b^2d^2h^2 = 0.$ Si angulus B rectus ponatur, paullo etiam ſimplicior efficitur, littera h in unitatem abeunte,

$$f^4 - \frac{2f^3\,d\,p}{m} + f^2\,\frac{(d^2\,p^2 - b^2)}{m^2} + \frac{2\,f\,d\,b^2\cos.\beta}{m^2} - \frac{b^2\,d^2}{m^2} = 0.$$

Si

Si angulus $ACD=\beta$ sit rectus, evanescente cosinu, quartus terminus evanescit, et aequatio mutatur in hanc: $f^4 - \frac{2f^3dn}{m} + \frac{f(d^2n^2-b^2h^2)}{m^2} - \frac{b^2d^2h^2}{m^2} = o$,

Coroll. V.

§. 159. Si angulus A sit duobus rectis aequalis, aequatio pro diagonali mutatur in hanc quadraticam, trapezio in triangulum BCD abeunte: $f^2(d^2 \mathit{sin}.\beta^2 - b^2h^2) + 2fdb^2h^2 \cos.\beta - b^2d^2h^2 = o$ ex qua operatione absoluta obtinetur haec aequatio:

$$f = \frac{bdh(-b\,\mathit{sin}.\lambda \cos.\beta \pm \mathit{sin}.\beta \sqrt{(d^2 - b^2 \mathit{sin}.\lambda^2)})}{d^2 \mathit{sin}.\beta^2 - b^2 \mathit{sin}.\lambda^2}$$

Solutionem continens, quae trigonometriae parum vel nihil cedit. Si ponatur angulus B aequalis duobus rectis, abeunte sic quadrilatero in triangulum ADC, aequatio mutatur in hanc quadraticam: $f^2 - \frac{2fdp}{m} + \frac{d^2p^2}{m^2} = o = \frac{(f^2 - dp)^2}{m}$ unde obtinetur: $fm = dp$, et $f: d = \mathit{sin}.(\beta+\psi): \mathit{sin}.\psi$ sive $d: \mathit{sin}.\psi = f: \mathit{sin}.(\beta+\psi)$, analogia trigonometrica, qua igitur aequatio tetragonometrica et operatio verificatur.

Coroll. VI.

§. 160. Si angulus A sit aequalis tribus rectis, cadente vertice A supra diagonalem BD, intra vel extra triangulum BCD, et prodeunte trapezio inverso, aequatio pro diagonali fit:

 f^4-

$f^4 - 2f^3\, d\, cos.\, \beta + f^2 (d^2 cos.\, \beta^2 - b^2 h^2) + 2 f d\, b^2 h^2$ $cos.\, \beta - b^2 d^2 h^2 = o$. Si vero angulus B ponatur aequalis tribus rectis, cadente B ad dextram diagonalis intra vel extra triangulum ADC, et trapezio inverso denuo prodeunte, habetur haec aequatio: $f^4 - \frac{2f^3 d\, p}{m} + \frac{f^2 (d^2 p^2 - b^2)}{m^2} +$

$$\frac{2 f d\, b^2 cos.\, \beta}{m^2} - \frac{b^2 d^2}{m^2} = o.$$

Coroll. VII.

§. 161. Si anguli A et ACD simul sumti aequentur duobus rectis, aequatio in hanc simpliciorem abit: $f^4 - \frac{f^2 b^2 h^2}{m^2} + \frac{2 f d\, b^2 h^2 cos.\, \beta -}{m^2}$

$- \frac{b^2 d^2 h^2}{m^2} = o$, et si in hac sumtione angulus uterque sumatur rectus aequatio fit: $f^4 - f^2 b^2 h^2 - b^2 d^2 h^2 = o$, ex qua obtinetur: $f = \frac{\sqrt{(b\, fin.\, \lambda\, (b\, fin.\, \lambda \pm \sqrt{b^2 fin.\, \lambda^2 + 4 d^2}))}}{\sqrt{2}}$

Quod si iidem anguli ponantur simul uni recto aequales, prodibit haec aequatio: $f^4 - \frac{2 f^3 d}{m} +$

$$\frac{f^2 (d^2 - b^2 h^2)}{m^2} + \frac{2 f\, b^2 d\, h^2}{m} - \frac{b^2 d^2 h^2}{m^2} = o.$$

Coroll. VIII.

§. 162. Si in aequatione primo inventa ponatur $f = b$, ut fiat $\sqrt{(f^2 - b^2 fin.\, \lambda^2)} = b$ $\sqrt{(1 - fin.\, \lambda^2)} = \pm\, b\, cos.\, \lambda$, aequatio mutatur in

in hanc ab irrationalitate liberam: $\pm d\, \textit{sin.} (\beta+\psi)$ $\textit{cos.}\, \lambda - - d\, \textit{cos.} (\beta+\psi)\, \textit{sin.}\, \lambda == b\, \textit{sin.}\, \psi\, \textit{cos.}\, \lambda -$ $- b\, \textit{sin.}\, \lambda\, \textit{cos.}\, \psi$ quae in hanc abit, adhibito signo affirmativo: $d\, \textit{sin.} (\beta+\psi-\lambda) == b\, \textit{sin.} (\psi-\lambda)$, sed negativo adhibito signo: $d\, \textit{sin.} (\beta+\lambda+\psi) =$ $= b\, \textit{sin.} (\lambda+\psi)$; in casu priore sequitur esse: $b : \textit{sin.} (\beta+\psi-\lambda) = d : \textit{sin.} (\psi-\lambda)$, at in posteriore: $b : \textit{sin.} (\beta+\lambda+\psi) == d : \textit{sin.} (\psi+\lambda)$. Prior analogia valet quamdiu angulus D est minor duobus rectis, sive quod eodem recidit, quamdiu angulus $ACD = \beta$, est affirmativus, quod si vero manentibus caeteris fiat D major duobus rectis, cadente D ultra diagonalem intra triangulum sinistrum, sive angulus ACD negativus, valet etiam prior aequatio paullulum immutata, ut sit: $d\, \textit{sin.} (\beta+\psi-\lambda) = b\, \textit{sin.} (\lambda-\psi)$. Idem casus etiam obtinet, si angulus B ultra diagonalem quomodocunque cadat. Sed analogia posterior vera non est, nisi angulus B fiat duobus rectis aequalis, quo in casu trapezium abit in triangulum dextrum, et analogia haec trigonometrica tunc obtinet: $b : \textit{sin.} (\beta+\psi) = d : \textit{sin.}\, \psi$.

Scholion III.

§. 163. Liquet ergo posito $b=f$, signum negativum *cos.* λ esse rejiciendum, cum enim triangulum ACB sit rectilineum, fieri non potest ut angulus uterque in hoc triangulo adjacens lateri AB, sit obtusus, $-\textit{cos.}\, \lambda$ angulum $-B$ obtusum indicante.

Scholion IV.

§. 164. In aequatione, quae in solutione problematis fuit inventa pro trapezio directo, non

aliae variationes occurrunt, quam quae ex ambiguitate ſigni radicalis et coſinus oriuntur. Quod ſi vero trapezium ita ſit partialiter inverſum, ut totum triangulum ADC cadat intra triangulum ABC, exiſtente angulo ACD majore quam angulo BAD, habetur haec aequatio:
$$(d \sin.(\beta-\psi) - f \sin.\psi)\sqrt{(f^2 - b^2 \sin.\lambda^2)} = -b \sin.\lambda(d \cos.(\beta-\psi) + f \cos.\psi).$$
Verum excedente angulo BAD angulum ACD, haec habetur:
$$(d \sin.(\psi-\beta) + f \sin.\psi)\sqrt{(f^2 - b^2 \sin.\lambda^2)} = -b \sin.\lambda(d \cos.(\psi-\beta) + f \cos.\psi).$$
Eaedem aequationes recurrunt, ſi trapezium ita ſit totaliter inverſum, ut latus AD cadat intra latus AB, latus autem CD, extra latus CB. Sin autem trapezium ita ſit totaliter inverſum ut latus AD cadat extra latus AB, latus autem CD intra latus CB, habetur haec aequatio:
$$(d \sin.(\beta+\psi) - f \sin.\psi)\sqrt{(f^2 - b^2 \sin.\lambda^2)} = b \sin.\lambda(-d \cos(\beta+\psi) + f \cos.\psi).$$
Quod ſi trapezium ita ſit partialiter inverſum ut totum triangulum ABC cadat intra triangulum ADC, eadem aequatio recurrit, neque praeterea aliae variationes occurrunt, niſi quae ex ambiguitate ſigni radicalis coſinuum oriri poſſunt. Generalis igitur aequatio erit hujusmodi:
$$(d \sin.(\beta\pm\psi) \mp f \sin.\psi)\sqrt{(f^2 - b^2 - \sin.\lambda^2)} = \pm b \sin.\lambda(\pm\lambda \cos.(\beta\pm\psi) \mp f \cos.\psi).$$

Problema XII.

Fig. XII. §. 165. In figura quadrilatera $ABCD$, inter haec ſex, latus $AB=a$, $BC=b$, $CD=d$, diagonalem $AC=f$, angulum $A=\psi$, et denique angulum $ACD=\beta$; aequationem invenire.

Solutio.

Solutio.

1) Demisso ab angulo B in diagonalem perpendiculo, cum sit per Elementa Geometriae segmentum $A\,b = \frac{a^2+f^2-b^2}{2f}$, erit perpendiculum

$$B\,b = \frac{\sqrt{(a^2 - (a^2+f^2-b^2)^2)}}{4f^2} = h \text{ brevitatis gratia,}$$

et in triangulo $B\,A\,C$ formo hanc analogiam: $1 : A\,B\,(a) = \text{fin.}\,B\,A\,C : h$, hinc sequitur esse:

$$\text{fin.}\,B\,A\,C = \frac{h}{a} \text{ et } \cos.\,B\,A\,C = \frac{\sqrt{(a^2-h^2)}}{a}.$$

2) Ex his et finu cofinuque anguli A formo finum et cofinum anguli $C\,A\,D$ et invenio:

$$\text{fin.}(\psi - CAB) = \frac{\text{fin.}\,\psi\sqrt{(a^2-h^2)} - h\cos.\,\psi}{a} = \text{fin.}\,CAD$$

$$\text{et } \cos.(\psi - CAB) = \frac{\cos.\,\psi\sqrt{(a^2-h^2)} + h\,\text{fin.}\,\psi}{a} = \cos.\,CAD.$$

3) Ex his et ex finu cofinuque anguli $A\,C\,D$ formo finum anguli D, qui igitur erit:

$$= \frac{\text{fin.}\,\beta\cos.\,\psi\sqrt{(a^2-h^2)} + h\,\text{fin.}\,\beta\,\text{fin.}\,\psi +}{a}$$

$$\frac{\cos.\,\beta\,\text{fin.}\,\psi\sqrt{(a^2-h^2)} - h\cos.\,\beta\cos.\,\psi}{a}$$

$$= \frac{\text{fin.}\,(\beta+\psi)\sqrt{(a^2-h^2)} - h\cos.\,(\beta+\psi)}{a} = \text{fin.}\,D.$$

4) His factis in triangulo dextro pervenio ad hanc analogiam: $A\,C : \text{fin.}\,D = CD : \text{fin.}\,CAD$, h. e.

$$f : \frac{\text{fin.}\,(\beta+\psi)\sqrt{(a^2-h^2)} - h\cos.\,(\beta+\psi)}{a} =$$

 $= d :$

$= d : \frac{\sin.\psi \sqrt{(a^2-h^2)} - h\cos.\psi}{a}$ unde conſequitur fore: $(d\sin.(\beta+\psi) - f\sin.\psi)\sqrt{(a^2-h^2)} =$ $= h(d\cos.(\beta+\psi) - f\cos.\psi)$. Q. E. I.

Coroll. I.

§. 166. Quia d non continetur in h, ex hac aequatione facillime invenitur latus DC cum ſit

$$d = \frac{f\sin.\psi\sqrt{(a^2-h^2)} - f\cos.\psi}{\sin.(\beta+\psi)\sqrt{(a^2-h^2)} - h\cos.(\beta+\psi)} = DC.$$

Sit ad abbreviandum $d\sin.(\beta+\psi) - f\sin.\psi = M$, $d\cos.(\beta+\psi) - f\cos.\psi = N$, et erit aequatio abbreviata: $M\sqrt{(a^2-h^2)} = hN$, ex qua ſubſtituto valore ipſius et operatione abſoluta, obtinetur haec aequatio pro latere

$$BC = b = \frac{\sqrt{((a^2+f^2)\sqrt{(M^2+N^2)} - 2afN)}}{\sqrt{(M^2+N^2)}}$$

ex eadem aequatione invenitur

$$\text{latus } AB = \frac{fN \pm \sqrt{(b^2(M^2+N^2) - M^2f^2)}}{\sqrt{(M^2+N^2)}} = a.$$

Coroll. II.

§. 167. Pro angulo A obtinendo habetur haec aequatio:

$$\text{tang.}\,\psi = \frac{(d\cos.\beta - f)h - d\sin.\beta\sqrt{(a^2-h^2)}}{hd\sin\beta + (d\cos.\beta - f)\sqrt{(a^2-h^2)}}$$

et angulus ipſe

$$\psi = \text{ang. tang.}\frac{(d\cos.\beta - f)h - d\sin.\beta\sqrt{(a^2-h^2)}}{hd\sin.\beta + (d\cos.\beta - f)\sqrt{(a^2-h^2)}}$$

Coroll.

Coroll. III.

§. 168. Poſito ad abbreviandum $cos.(\beta+\psi) = q$, $cos.\psi = n$, aequatio pro diagonali ad ſextum gradum aſcendit et eſt hujusmodi:

$$\begin{array}{l} f^6 - 2f^5 d\,cos.\beta + 2(a^2-b^2)\,f^4 - 4d\,cos.\beta(a^2-b^2) \\ \qquad\qquad\qquad\; -4a^2n^2 \qquad\qquad\quad +8a^2d^2nq \\ \qquad\qquad\qquad\; +\,d^2 \end{array}$$

$$\begin{array}{l} f^3 + 2d^2(a^2-b^2)\,f^2 - 2fd\,cos.\beta(a^2-b^2) + d^2(a^2-b^2) = 0. \\ \quad +(a^2-b^2)^2 \\ \quad -4a^2d^2q^2 \end{array}$$

Sed in eo caſu, quo AB aequatur BC, ut ſit $a=b$ et triangulum ABC aequicrurum, aequatio multum contrahitur, et ad quartum gradum deſcendit, ut fiat: $f^4 - 2f^3 d\,cos.\beta + f^2(d^2-4a^2n^2) + 8fa^2dn\,q - 4a^2d^2q^2 = 0.$

Scholion I.

§. 169. Inter ſex illos caſus, qui in aequatione continentur, non datur niſi unus Tetragonometriae proprius, qui tunc obtinet, quando ex diagonali quaeritur conſtructio figurae, ſed dolendum aequationem hanc eſſe ſexti gradus et nimis complicatam, ut ad praxin ſit parum utilis, cum tamen hic caſus problema ſuppeditet in Geometria practica novum utile ſimul et ſatis pulchrum, caeteri caſus quinque utrique methodo et trigonometricae et tetragonometricae ſubſunt, ſed ita ut trigonometrice facilius brevius et concinnius ſolvantur, quorum quatuor evolvi, quia ex aequatione haud ita operoſe conſequuntur. Quintum conſulto omiſi, qui obtinet pro inveniendo angulo ACD, qui ex aequatione operoſius evolvitur, et per aequationem magis complicatam exprimitur.

Coroll.

Coroll. IV.

§. 170. Si ponatur angulus A aequalis duobus rectis, trapezio in triangulum BCD abeunte, aequatio paululum mutatur in iis terminis, in quibus occurrunt litterae n et q, cum fiat prior $=-1$, posterior $=-\cos.\beta$, et continet solutionem problematis novi de triangulis hujusmodi: Datis in triangulo BCD lateribus BC, CD, et parte lateris tertii BE, et angulo ECD, parte anguli C, invenire rectam AC, et resolvere triangulum tum totale tum partiale. Si in hoc casu sit AB aequalis BC, aequatio quarti gradus prius inventa paululum immutatur, ut fiat: $f^4-2f^3d\cos.\beta+f^2(d^2-4a^2)+8fa^2d\cos.\beta-4a^2d^2\cos.\beta^2=0$.

Coroll. V.

§. 171. Quod si angulus ACD evanescere ponatur, abeunte trapezio in triangulum ABC, aequatio primo inventa, in hanc mutatur: $\sin.\psi\sqrt{(a^2-h^2)}=h\cos.\psi$, sive $\tang.\psi\sqrt{(a^2-h^2)}=h$ et brevitatis causa posito $\tang.\psi=t$, absoluta operatione habetur: $f=\frac{a\pm\sqrt{(b^2(t^2+1)-a^2t^2)}}{\sqrt{(t^2+1)}}$ sive quoniam $\sqrt{(t^2+1)}$ est secans anguli A, erit $f=\frac{a\pm\sqrt{(b^2\sec.\psi^2-a^2\tang.\psi^2)}}{\sec.\psi}$, quae in hanc mutatur: $f=a\cos.\psi\pm\sqrt{(b^2-a^2\sin.\psi^2)}$.

Scholion II.

§. 172. Suppeditat haec aequatio novam solutionem trianguli ABC, nam in eo casu quo ex

ex datis lateribus AB, BC, et angulo BAC quaerendum eſt latus tertium AC, ad idem communi methodo ſolvendum, duabus analogiis opus eſt; tamen methodo tetragonometrica haec ſolutio facile obtinetur, nam demiſſo ab angulo B, in latus quaeſitum perpendiculo Bb, ſtatim apparet ſegmentum Ab eſſe $= a \cos. \psi$ et ſegmentum alterum $bc = \surd(b^2 - a^2 \sin. \psi^2)$ et conſequenter fore in caſu anguli ACB acuti latus $f = a \cos. \psi + \surd(b^2 - a^2 \sin. \psi^2)$, ſed in caſu ejusdem obtuſi obtinet ſignum negativum, ut ſit $f = a \cos. \psi - \surd(b^2 - a^2 \sin. \psi)$. Verum ex aequatione generali hic caſus particularis operoſius elicitur.

Coroll. VI.

§. 173. Si angulus A ponatur aequalis tribus rectis, cadit vertex A ſupra diagonalem BD intra vel extra triangulum BCD, et prodit trapezium vel ex parte vel totaliter inverſum, ſed aequatio generalis ſit paullo ſimplicior, cum duo termini, in quibus occurrit $n = \cos. \psi$ evaneſcant. Sed in hoc caſu, ſi ſit $AB = BC$; aequatio quarti gradus ſit etiam paullo ſimplicior talis: $f^4 - 2f^3 d \cos. \beta + f^2 d^2 - 4a^2 d^2 \sin. \beta^2 = 0.$

Scholion III.

§. 174. In aequatione, quae in reſolutione problematis fuit inventa pro trapezio directo, aliae variationes non occurrunt, quam quae ex ambiguitate ſigni radicalis et coſinus oriuntur, et in littera h, prout perpendiculum cadit infra vel extra figuram. Quodſi vero trapezium ita

ſit

ſit partialiter inverſum, ut totum triangulum ACD cadat intra triangulum ABC, exiſtente angulo ACD majore, quam angulo BAD, habetur haec aequatio: $(d\,ſin.(\beta - \psi) + f\,ſin.\,\psi)\sqrt{(a^2-h^2)} = h(-d\,cos.(\beta-\psi)+f\,cos.\,\psi)$. Sed excedente angulo BAD angulum ACD, haec habetur: $(-d\,ſin.(\psi-\beta)+f\,ſin.\,\psi)\sqrt{(a^2-h^2)} = h(-d\,cos.(\psi-\beta)+f\,cos.\,\psi)$. Si trapezium ita ſit totaliter inverſum, ut latus AD cadat intra latus AB, CD vero extra CB, eaedem aequationes recurrunt. Si trapezium ita ſit totaliter inverſum, ut latus AD cadat extra latus AB, et latus CD intra latus CB, obtinet haec aequatio: $(d\,ſin.(\beta+\psi) - f\,ſin.\,\psi)\sqrt{(a^2-h^2)} = h(-cos.(\beta+\psi)+f\,cos.\,\psi)$. Eadem aequatio recurrit, ſi totum triangulum ABC cadat intra triangulum ADC. Praeterea nullae variationes occurrunt, niſi quae ex ambiguitate ſigni radicalis et coſinus naturae oriri poſſunt, quare generalis aequatio erit talis pro omni trapezio tam directo quam inverſo:

$$\left(\pm d\,ſin.\binom{\beta\pm\psi}{\psi-\beta} \mp f\,ſin.\,\psi\right)\sqrt{(a^2-h^2)} = h\left(\pm d\,cos.\binom{\beta\pm\psi}{\psi-\beta} \mp f\,cos.\,\psi\right).$$

Problema XIII.

Fig. XIII. §. 175. In figura quadrilatera $ABCD$, inter haec ſex: latus $AB = a$, $CD = d$, diagonalem $AC = f$, et angulum $A = \psi$, $B = \lambda$ et $ACD = \beta$; aequationem invenire.

Solutio.

Solutio.

Cum ſit in ſolutione ſexti problematis (§. 68.) inventus $fin. CAD = \frac{-fin.(\lambda+\psi) \surd (f^2 - a^2 fin. \lambda^2)}{f}$ $\frac{-a\, fin. \lambda\, cos. (\lambda+\psi)}{f}$, et praeterea inveniatur eſſe $cos.\ CAD = \frac{-cos.(\lambda+\psi) \surd (f^2 - a^2 fin. \lambda^2)}{f}$ $\frac{+a\, fin. \lambda\, fin. (\lambda+\psi)}{f}$, et ex ſinu coſinuque anguli ACD formatur ſinus anguli D, qui igitur invenitur eſſe: $= \frac{-fin. \beta\, cos. (\lambda+\psi) \surd (f^2 - a^2 fin. \lambda^2)}{f}$ $\frac{-cos.\ \beta\ fin.\ (\lambda+\psi) \surd (f^2 - a^2 fin.\ \lambda^2)}{f}$ $+ \frac{a\, fin. \lambda\, fin. \beta\, fin. (\lambda+\psi) - a\, fin. \lambda\, cos. \beta\, cos. (\lambda+\psi)}{f}$ $= fin. D = \frac{-fin. (\beta+\lambda+\psi) \surd (f^2 - a^2 fin. \lambda^2)}{f}$ $\frac{-a\, fin.\ \lambda\ cos.\ (\beta+\lambda+\psi)}{f}$, in triangulo dextro pervenio ſtatim ad hanc analogiam: $AC : fin.\ D = CD : fin.\ CAD$, i. e. $f : \frac{-fin.\ (\beta+\lambda+\psi) \surd (f^2 - a^2 fin.\ \lambda^2)}{f}$ $\frac{-a\ fin.\ d\ cos.\ (\beta+\lambda+\psi)}{f} =$ $d : \frac{-fin. (\lambda+\psi) \surd (f^2 - a^2 fin. \lambda^2) - a\, fin. \lambda\, cos. (\lambda+\psi)}{f}$,

et

et consequenter habetur haec aequatio:
$(f\,\mathit{fin.}\,(\lambda+\psi) - d\,\mathit{fin.}\,(\beta+\lambda+\psi))\sqrt{}(f^2-a^2\,\mathit{fin.}\,\lambda^2)$
$= a\,\mathit{fin.}\,\lambda\,(d\,\mathit{cos.}\,(\beta+\lambda+\psi) - f\,\mathit{cos.}\,(\lambda+\psi))$
Q. E. I.

Scholion I.

§. 176. Formato ut prius finu anguli CAD, et formatis ut supra Probl. VI. (§. 68.) finu et cofinu anguli ACB, cum habeantur nomina primitiva angulorum A, B, ACD, erit nomen derivativum fummae angulorum reliquorum $ACB+D = 2\pi - \beta - \lambda - \psi$, et confequenter $\mathit{fin.}\,(ACB+D) = -\mathit{fin.}\,(\beta+\lambda+\psi$ et $\mathit{cos.}\,ACB+D) = \mathit{cos.}\,(\beta+\lambda+\psi)$, quare ex his et ex finu cofinuque anguli ACB, formo $\mathit{fin.}\,D = \mathit{fin.}\,(ACB+D-ACB^2)$

$$= \frac{-\mathit{fin.}\,(\beta+\lambda+\psi)\sqrt{}(f^2 - a^2\,\mathit{fin.}\,\lambda^2)}{f}$$

$$\frac{-a\,\mathit{fin.}\,\lambda\,\mathit{cos.}\,(\beta+\lambda+\psi)}{f}$$ et factis reliquis, ut fupra, ad eandem aequationem pervenio.

Coroll. I.

§. 177. Ex hac aequatione facillime obtinetur valor lateris CD per hanc aequationem:

$$d = \frac{f(a\,\mathit{fin.}\,\lambda\,\mathit{cos.}(\lambda+\psi)+\mathit{fin.}(\lambda+\psi)\sqrt{}(f^2-a^2\,\mathit{fin.}\,\lambda^2))}{a\,\mathit{fin.}\,\lambda\,\mathit{cos.}(\beta+\lambda+\psi)+\mathit{fin.}(\beta+\lambda+\psi)\sqrt{}(f^2-a^2\,\mathit{fin.}\,\lambda^2)}$$

et lateris AB per hanc alteram:

$$a = \frac{(f\,\mathit{fin.}\,(\lambda+\psi) - d\,\mathit{fin.}\,(\beta+\lambda+\psi))f}{\mathit{fin.}\,\lambda\sqrt{}(f^2+d^2 \pm 2fd\,\mathit{cos.}\,\beta)}.$$

Coroll.

Coroll. II.

§. 178. Pro angulo A inveniendo habetur haec aequatio:

$tang.\ (\lambda + \psi) =$

$$= \frac{a\,fin.\lambda\,(f - d\,cos.\,\beta) - d\,fin.\,\beta\,\sqrt{(f^2 - a^2\,fin\,\lambda^2)}}{(d\,cos.\,\beta - f)\sqrt{(f^2 - a^2\,fin.\,\lambda^2)} - a\,d\,fin.\,\lambda\,fin.\,\beta}$$

et consequenter ipse angulus hoc modo exprimitur:

$\psi = ang.\ tang.$

$$\frac{a\,fin.\lambda\,(f - d\,cos.\,\beta) - d\,fin.\,\beta\,\sqrt{(f^2 - a^2\,fin.\lambda^2)}}{(d\,cos.\,\beta - f)\sqrt{(f^2 - a^2\,fin.\,\lambda^2)} - a\,d\,fin.\,\lambda\,fin.\,\beta} - \lambda.$$

Coroll. III.

§. 179. Posito ad abbreviandum $tang.\,(\lambda + \psi) = t$, $\sqrt{(f^2 - a^2\,fin.\,\lambda^2)} = M$, $fin.\lambda = h$, $fin.\,\beta = x$, $cos.\ \beta = \sqrt{(1 - x^2)}$, substitutionibus et transpositionibus factis, erit aequatio abbreviata: $(Mt + a\,h)\sqrt{(1 - x^2)} = (Mt + a\,h)\,f - (M - a\,h\,t)x$, et rursus ad abbreviandum posito. $Mt + a\,h = P$, $M - a\,h\,t = Q$, erit aequatio abbreviata $P\sqrt{(1 - x^2)} = Pf - Q\,x$, ex qua operatione absoluta obtinetur haec aequatio: $x = \frac{fPQ \pm P\sqrt{(P^2 + Q^2 - f^2P^2)}}{P^2 + Q^2} =$

$$= \frac{P\,(fQ \pm \sqrt{(Q^2 + P^2(1 - f^2))})}{P^2 + Q^2} = fin.\ \beta.$$

Coroll. IV.

§. 180. Ex corollarii primi aequatione pro valore lateris $A\,B$ inventa, eliminatis secundis dimensionibus anguli $B = \lambda$, apparet in numeratore et denominatore unam dimensionem inesse

et confequenter pro angulo B aequationem fimplicem tangentialem inveniri poffe, quae finuum feparatione et reductione facta invenitur effe: $tang.\, \lambda =$

$$= \frac{(f\, fin.\, \psi - d\, fin.\, (\beta + \psi) f}{a \surd (f^2 + d^2 \pm 2 f d \cos.\beta) - (f \cos.\psi - d \cos.(\beta + \psi)) f}$$

Coroll. V.

§. 181. Si in aequatione primi corollarii ponatur ad abbreviandum $fin.\, \lambda = h$, $fin.\, (\lambda + \psi) = m$, $fin.\, (\beta + \lambda + \psi) = p$, erit factis fubftitutionibus $a = \frac{f(fm - dp)}{h \surd (f^2 + d^2 - 2 f d \cos.\beta)}$, ex qua operatione abfoluta obtinetur pro diagonali haec aequatio quarti gradus: $f^4 - \frac{2 f^3 d p}{m} + f^2 \frac{(d^2 p^2 - a^2 h^2)}{m^2} \pm \frac{2 a^2 f d\, h^2 \cos.\beta}{m} - \frac{m^2 d^2 h^2}{m^2} = 0.$

Scholion II.

§. 182. Ita fex cafus, qui in aequatione continentur funt evoluti, quorum ultimus unicus Tetragonometriae proprius, fuppeditat problema in Geometria practica novum utile, juxta ac fatis pulchrum, quod trigonometrice minime folvitur. Caeteri quinque cafus utrique methodo et trigonometricae, et tetragonometricae fubfunt, fed ita ut trigonometrice fere facilius et expeditius folvantur, quos ideo evolvi, quia haud difficulter ex aequatione eruuntur.

Coroll.

Coroll. VI.

§. 183. Si anguli A et B ponantur aequales duobus rectis, ut prodeat trapezium parallelarum basium AD, BC, aequatio in solutione inventa mutatur in hanc: $d\,\text{fin.}\,\beta\, \sqrt{(f^2 - a^2\,\text{fin.}\,\lambda^2)} = a\,\text{fin.}\,\lambda\,(f - d\cos.\beta)$, vel etiam hanc: $d\,\text{fin.}\,\beta\, \sqrt{(f^2 - a^2\,\text{fin.}\,\psi^2)} = a\,\text{fin.}\,\psi\,(f - d\cos.\beta)$. Unde obtinetur haec analogia: $a\,\text{fin.}\,\lambda : \sqrt{(f^2 - a^2\,\text{fin.}\,d^2)} = d\,\text{fin.}\,\beta : f - d\cos.\beta$, quae hoc suppeditat

Theorema.

In trapezio parallelarum basium est ut perpendiculum in triangulo finistro ABC, ab angulo A in unam parallelarum BC demissum ad rectam Cb, inter se et latus dextrum interceptum, ita perpendiculum ab angulo D trianguli dextri in diagonalem demissum, ad partem diagonalis Ad inter angulum A et se interceptum, unde statim sequitur triangula AbC et ADd esse similia.

Scholion III.

§. 184. Hujus theorematis demonstratio synthetica statim apparet. Cum enim per hypothesin sint parallela latera AD, BC, erit angulus DAd aequalis angulo ACb. Sed ob normales Ab, Dd erunt anguli AbC, AdD recti, et consequenter aequales, cum igitur triangula AbC, AdD sint rectangula et acutangula, tertius angulus tertio aequatur, et consequenter habent latera homologa proportionalia, et similia sunt.

Coroll. VII.

§. 185. Cum ad abbreviandum positum sit: $\text{fin.}(\lambda+\psi)=m$, et $\text{fin.}(\beta+\lambda+\psi)=p$, $\text{fin.}\lambda=h$; in casu corollarii sexti aequatio pro diagonali in hanc mutatur: $f^2(d^2\,\text{fin.}\,\beta^2 - a^2\,\text{fin.}\,d^2) \pm 2a^2fd\,\text{fin.}\,\lambda^2\cos.\beta - a^2d^2\,\text{fin.}\,d^2 = o$, unde operatione ad finem perducta elicitur pro diagonali in casu parallelismi laterum AD, BC, haec aequatio quadratica impura.

$$f=\frac{ad\,\text{fin.}\lambda(\mp a\,\text{fin.}\lambda\cos.\beta\pm\text{fin.}\beta\sqrt{(d^2-a^2\text{fin.}\lambda^2)})}{(d\,\text{fin.}\beta+a\,\text{fin.}\lambda)\;(d\,\text{fin.}\beta-a\,\text{fin.}\lambda)}$$

Coroll. VIII.

§. 186. Si angulus A sit duobus rectis aequalis, lateribus AB, AD, in diagonalem BD incidentibus, et quadrilatero in triangulum BCD abeunte, aequatio pro diagonali in hanc mutatur:

$$f^4-\frac{2f^3d\,\text{fin.}(\beta+\lambda)}{\text{fin.}\lambda}+f^2\left(\frac{d^2\,\text{fin.}(\beta+\lambda^2)}{\text{fin.}\lambda^2}-a^2\right)$$

$\pm 2fa^2d\cos.\beta - a^2d = o$. Si vero angulus B ponatur aequalis duobus rectis, lateribus AB, BC in diagonalem alteram AC incidentibus, trapezio iterum in triangulum ADC abeunte, aequatio pro diagonali abit in hanc: $f^2-\frac{2fd\,\text{fin.}(\beta+\psi)}{\text{fin.}\psi}+$

$$\frac{d^2\,\text{fin.}(\beta+\psi)^2}{\text{fin.}\psi^2}=\frac{(f-d\,\text{fin.}\beta+\psi)^2}{\text{fin.}\psi^2}.$$

Unde elicitur $f\,\text{fin.}\psi=d\,\text{fin.}(\beta+\psi)$ et consequenter habetur haec analogia trigonometrica: $f:\text{fin.}(\beta+\psi)=d:\text{fin.}\psi$, qua igitur aequationes et operationes verificantur.

Schol.

Scholion IV.

§. 187. Aequatio pro caſu anguli A duobus rectis aequalis, novum prorſus ad triangula ſpectans ſolvit problema hujusmodi:

Datis in triangulo BCD, angulo B, latere CD, angulo partiali ECD, et parte EB lateris BD; invenire EC, et reſolvere triangula, quod problema alia methodo quam tetragonometrica non ſolvitur.

Coroll. IX.

§. 188. Si angulus A ponatur aequalis tribus rectis, vertice ſuper diagonalem intra vel extra triangulum BCD cadente, et trapezio inverſo prodeunte, et ad aequationem abbreviandam fiat $cos. \lambda = k$, $cos. (\beta + \lambda) = n$, $tang. \lambda = t$, fiet aequatio pro diagonali: $f^4 - \frac{2 f^3 d n}{k}$

$$+ \frac{f^2 (d^2 n^2 - a^2 t^2)}{k^2} \pm 2 f a^2 d t^2 cos. \beta - a^2 d^2 t^2 = 0$$

Si vero angulus B ſit aequalis tribus rectis, cadente vertice B ad dextram diagonalis AC, intra vel extra triangulum ADC, trapezio inverſo denuo prodeunte, poſito ad aequationem abbreviandam $cos. (\beta + \psi) = q$, $cos. \psi = r$, aequatio pro diagonali abit in hanc: $f^4 - \frac{2 f^3 d q}{r} +$

$$\frac{f^2 (d^2 q^2 - a^2)}{r^2} \pm \frac{2 f a^2 d \, cos. \beta}{r^2} - \frac{a^2 d^2}{r^2} = 0.$$

Coroll. X.

§. 189. Si tres anguli ſimul ſumti A, B, ACD ſint duobus rectis aequales, aequa-

tio pro diagonali in hanc mutatur:

$$\frac{f^4 - f^2 a^2 h^2 \pm 2 f a^2 d h^2 \cot. \beta}{\sin. \beta^2} - \frac{a^2 d^2 h^2}{\sin. \beta^2} = 0$$

Si angulus B, ponatur rectus, prodit rurſus aequatio poſterior corollarii praecedentis. Si vero angulus A ponatur rectus, prodit rurſus aequatio prior corollarii praecedentis. Si denique angulus ACD ponatur rectus prodibit haec aequatio:

$$f^4 - 2f^3 d \cot. (\lambda + \psi) + f^2 \left(d^2 \operatorname{cotang.} (\lambda + \psi)^2 - \frac{a^2 h^2}{m^2}\right) - \frac{a^2 d^2 h^2}{m^2} = 0.$$

Coroll. XI.

§. 190. Quod ſi angulus ACD ſit nihilo aequalis, ut ex trapezio fiat triangulum ſiniſtrum ABC, aequatio pro diagonali in hanc mutatur:

$$f^4 - 2f^3 d + f^2 \left(d^2 - \frac{a^2 h^2}{m^2}\right) \pm \frac{2 f a^2 d h^2}{m^2} - \frac{a^2 d^2 h^2}{m^2} = 0$$

quae per diviſorem quadratum $f^2 - 2fd + d^2$, diviſa, dat in quotiente $f^2 - \frac{a^2 h^2}{m^2} = 0$, unde ſit $f \sin. (\lambda + \psi) = a \sin. \lambda$, et analogia trigonometrica $f : \sin. \lambda = a : \sin. (\lambda + \psi)$ quo ipſo operationes verificantur. Si vero angulus ACD ſit negativus, cadente latere CD ad ſiniſtram diagonalis intra vel extra triangulum ABC, et trapezio inverſo prodeunte, aequatio (§. 175.) mutatur in hanc: $(f \sin. (\lambda + \psi) - d \sin. (\lambda + \psi - \beta))$ $\sqrt{(f^2 - a^2 \sin. \lambda^2)} = a \sin. \lambda \; (d \cos. (\lambda + \psi - \beta) - f \cos. (\lambda + \psi))$. Sed aequatio pro diagonali coroll. V. manet invariata, niſi quod in quantitate per p expreſſa pro $+\beta$ ſubſtituendum ſit $-\beta$.

Schol.

Scholion V.

§. 191. Aequatio hujus problematis in reſolutione inventa pro trapezio directo valet in hypotheſi figurae conſtructae, nimirum, quando ſumma angulorum B et ACB excedit rectum. Si vero minor recto ſit, et angulus BAD excedat angulum B; valet haec aequatio:
$(f\,\text{ſin.}(\psi-\lambda)-d\,\text{ſin.}(\beta+\psi-\lambda))\sqrt{(f^2-a^2\,\text{ſin.}\lambda^2)}$
$=a\,\text{ſin.}\,\lambda\,(f\cos.(\psi-\lambda)-d\cos.(\beta+\psi-\lambda))$;
Sin autem angulus B major ſit angulo A, tamen minor ſumma angulorum A et ACD, locum habet haec aequatio:
$(f\,\text{ſin.}(\lambda-\psi)+d\,\text{ſin.}(\beta+\psi-\lambda))\sqrt{(f^2-a^2\,\text{ſin.}\lambda^2)}$
$=a\,\text{ſin.}\,\lambda(-f\cos.(\lambda-\psi)+d\cos.(\beta+\psi-\lambda))$;
Si vero etiam eandem excedat, haec habetur:
$(f\,\text{ſin.}(\lambda-\psi)-d\,\text{ſin.}\,\lambda-\beta-\psi))\sqrt{(f^2-a^2\,\text{ſin.}\lambda^2)}$
$=a\,\text{ſin.}\,\lambda\,(-f\cos.(\lambda-\psi)+d\cos.(\lambda-\beta-\psi))$,
Aequatio igitur pro trapezio directo generalis eſt hujusmodi:

$$\left(f\,\text{ſin.}\begin{pmatrix}\lambda+\psi\\ \psi-\lambda\end{pmatrix}\mp d\,\text{ſin.}\begin{pmatrix}\beta+\psi\pm\lambda\\ \lambda-\psi-\beta\end{pmatrix}\right)\sqrt{(f^2-a^2\,\text{ſin.}\lambda^2)}$$

$$=a\,\text{ſin.}\lambda\left(\pm f\cos.\begin{pmatrix}\lambda\pm\psi\\ \psi-\lambda\end{pmatrix}\mp d\cos.\begin{pmatrix}\beta+\psi\pm\lambda\\ \lambda-\psi-\beta\end{pmatrix}\right).$$

Si trapezium ſit partialiter inverſum ita, ut totum triangulum ADC, cadat intra triangulum ABC, exiſtente ſumma angulorum B et ACB recto majore et ſumma angulorum A et B excedente angulum ACD; habetur haec aequatio:
$(f\,\text{ſin.}(\lambda+\psi)+d\,\text{ſin.}(\lambda+\psi-\beta))\sqrt{(f^2-a^2\,\text{ſin.}\lambda^2)}$
$=a\,\text{ſin.}\,\lambda\,(-f\cos.(\lambda+\psi)+d\cos.(\lambda+\psi-\beta))$
ſed excedente angulo ACD ſummam angulorum A et B, habetur haec caeteris manentibus iisdem:

$(f\,\text{ſin.}\,(\lambda+\psi) - d\,\text{ſin.}\,(\beta-\lambda-\psi))\,\sqrt{(f^2-a^2\,\text{ſin.}\,\lambda^2)}$
$= a\,\text{ſin.}\,\lambda\,(-f\,\text{cos.}\,(\lambda+\psi) + d\,\text{cos.}\,(\beta-\lambda-\psi))$.
Si ſumma angulorum *B* et *ACB* ſit recto minor, et angulus *A* minor angulo *B*, habetur haec aequatio:
$(f\,\text{ſin.}\,(\lambda-\psi) - d\,\text{ſin.}\,(\beta+\lambda-\psi))\,\sqrt{(f^2-a^2\,\text{ſin.}\,\lambda^2)}$
$= a\,\text{ſin.}\,\lambda\,(-f\,\text{cos.}\,(\lambda-\psi) + d\,\text{cos.}\,(\beta+\lambda-\psi))$.
Sin autem caeteris manentibus iisdem, angulus *A* excedit angulum *B*; habetur haec:
$(f\,\text{ſin.}\,(\psi-\lambda) + d\,\text{ſin.}\,(\beta+\lambda-\psi))\,\sqrt{(f^2-a^2\,\text{ſin.}\,\lambda^2)}$
$= a\,\text{ſin.}\,\lambda\,(f\,\text{cos.}\,(\psi-\lambda) - d\,\text{cos.}\,(\beta+\lambda-\psi))$.
Quod ſi vero etiam excedat ſummam angulorum *B* et *A C D*; habetur haec:
$(f\,\text{ſin.}\,(\psi-\lambda) - d\,\text{ſin.}\,(\psi-\beta-\lambda))\,\sqrt{(f^2-a^2\,\text{ſin.}\,\lambda^2)}$
$= a\,\text{ſin.}\,\lambda\,(f\,\text{cos.}\,(\psi-\lambda) - d\,\text{cos.}\,(\psi-\beta-\lambda))$
Si trapezium ita ſit totaliter inverſum, ut latus *AD* cadat intra latus *AB*; latus *CD* extra latus *CB*, eaedem aequationes recurrunt. Si trapezium ita ſit totaliter inverſum, ut latus *A D* cadat extra latus *A B*, latus autem *C D* extra latus *CB*; prodit haec aequatio, excedente ſumma angulorum *B* et *A C B* rectum:
$(-f\,\text{ſin.}\,(\lambda+\psi) + d\,\text{ſin.}\,(\beta+\lambda+\psi))\,\sqrt{(f^2-a^2\,\text{ſin.}\,\lambda^2)}$
$= a\,\text{ſin.}\,\lambda\,(f\,\text{cos.}\,(\lambda+\psi) - d\,\text{cos.}\,(\beta+\lambda+\psi))$,
quae non niſi ſignis omnibus contrariis differt ab aequatione, quae in reſolutione problematis fuit inventa. Quod ſi vero eadem ſumma ſit recto minor, reproducitur aequatio hujus ſcholii prima, ſiquidem angulus *A* excedat angulum *B*, ſin autem angulus *B* excedat angulum *A*, minor autem exiſtat ſumma angulorum *A* et *ACD*, reproducitur hujus ſcholii aequatio ſecunda, cum ſignis contrariis. Quod ſi vero angulus *B* excedat non modo angulum *A*, ſed etiam ſummam angulorum *A* et *ACD*, reproducitur hujus ſcholii aequa-

aequatio tertia cum ſignis contrariis. Si trapezium ita ſit totaliter inverſum, ut totum triangulum ABC cadat intra triangulum ADC, eaedem aequationes reproducentur. Igitur generaliſſima pro omni trapezio tam directo quam inverſo, aequatio aliter exprimi nequit, quam hoc modo:

$$\left(\pm f\,\text{ſin.}\left(\begin{matrix}\lambda\pm\psi\\ \psi-\lambda\end{matrix}\right)\mp d\,\text{ſin.}(\pm\beta\pm\psi\pm\lambda)\right)\sqrt{(y^2-a^2\,\text{ſin.}\lambda^2)}$$

$$= a\,\text{ſin.}\lambda\left(\mp f\cos.\left(\begin{matrix}\lambda\pm\psi\\ \psi-\lambda\end{matrix}\right)\mp d\cos.(\pm\beta\pm\psi\pm\lambda)\right).$$

Problema XIV.

§. 192. In figura quadrilatera $ABCD$, inter haec ſex: latus $AD=c$, $BC=b$, diagonalem $AC=f$, angulum $A=\psi$. $B=\lambda$, $ACD=\beta$; aequationem invenire. Fig. XIV.

Solutio.

1) In triangulo dextro formo hanc analogiam $AD\,(c):\ \text{ſin.}\ ACD\,(\text{ſin.}\beta) = AC\,(f):$ $\text{ſin.}\,D = \frac{f\,\text{ſin.}\beta}{c}$ et hinc erit $\cos.D = \frac{\sqrt{(c^2-f^2\,\text{ſin.}\beta^2)}}{c}$

2) Ex his et ex ſinu coſinuque anguli ACD, formo ſinum et coſinum anguli tertii CAD et obtineo $\text{ſin.}\,CAD = \frac{\text{ſin.}\beta\sqrt{(c^2-f^2\,\text{ſin.}\beta^2)}+f^2\,\text{ſin.}\beta\cos.\beta}{c}$,

nec non $\cos.CAD = \frac{-\cos.\beta\sqrt{(c^2-f^2\,\text{ſin.}\beta^2)}+f\,\text{ſin.}\beta^2}{c}$

3) Ex his et ex ſinu coſinuque anguli A formo ſinum anguli CAB in triangulo ſiniſtro. Cum enim ſit $CAB = A - CAD = \psi - CAD$; habebitur:

$fin.\ (\psi - CAD) =$

$$= \frac{-fin.\psi\, cos.\beta\, \surd(c^2-f^2 fin.\beta^2)+f\, fin.\psi\, fin.\beta^2}{c}$$

$$- \frac{cos.\psi\, fin.\beta\, \surd(c^2-f^2 fin.\beta^2)-f\, fin\,\beta\, cos.\beta\, cos.\psi}{c}$$

$$= \frac{-fin.(\beta+\psi)\surd(c^2-f^2 fin.\beta^2)-f\, fin.\beta\, cos.(\beta+\psi)}{c}$$

$= fin.\ CAB.$

4) Hinc in triangulo finiftro pervenio ad hanc analogiam: $AC: fin.\ B = BC: fin.\ CAB$, h. e. $f: fin.\lambda = b: \frac{-fin.(\beta+\psi)}{c} \cdot \frac{\surd(c^2-f^2 fin.\beta^2)-f fin.\beta\, cos.(\beta+\psi)}{c}$ unde fit: $bc\, fin.\lambda = -f\, fin.(\beta+\psi)\surd(c^2-f^2 fin.\beta^2)- -f^2 fin.\beta\, cos.(\beta+\psi)$ Q. E. I.

Scholion I.

§. 193. In folutione problematis XI. (§. 152.) inventus fuit finus anguli $D = \frac{fin.(\beta+\psi)\surd(f^2-b^2 fin.\lambda^2)-b\, fin.\lambda\, cos.(\beta+\psi)}{f}$ ac proinde in triangulo dextro ftatim pervenio ad hanc analogiam: $AC: fin.\ D = AD: fin.\ ACD$, h. e. fymbolis fubftitutis: $f: \frac{fin.(\beta+\psi)\surd(f^2-b^2 fin.\lambda^2)-b\, fin.\lambda\, cos.(\beta+\psi)}{f} = = c: fin.\beta$, et confequenter habetur haec aequatio: $f^2 fin.\beta = c\, fin.(\beta+\psi)\surd(f^2-b^2 fin.\lambda^2)- - bc\, fin.\lambda\, cos.(\beta+\psi)$. Haec aequatio a priore formaliter quidem, fed non realiter differt, cum

ſi ex hac quaeras valorem $b\,c\,\mathit{ſin}.\,\lambda$ ad illam pervenias, et contra, ſi ex illa quaeras valorem $f^2\,\mathit{ſin}.\,\beta$ ad hanc pervenias, conſequenter prior eſt poſterioris una radicum quadraticarum, cujus ſignum radicale negativum, et poſterior prioris una radicum quadraticarum adhibito ſigno radicali affirmativo. Prior igitur aequatio ſigno affirmativo adjecto generalius ſcribitur hoc modo: $b\,c\,\mathit{ſin}.\,\lambda = \pm f\,\mathit{ſin}.\,(\beta+\psi)\surd(c^2 - f^2\,\mathit{ſin}.\,\beta^2) - f^2\,\mathit{ſin}.\,\beta\,\cos.\,(\beta+\psi)$, cum ſic radix utraque poſterioris repraeſentetur diſtinctius; et poſterior generalius exhibetur hoc modo, ſigno negativo adjecto: $f^2\,\mathit{ſin}.\,\beta = \pm\,\mathit{ſin}.\,(\beta+\psi)\surd(f^2 - b^2\,\mathit{ſin}.\,\lambda^2) - b\,c\,\mathit{ſin}.\,\lambda\,\cos.\,(\beta+\psi)$, cum ſic radices aequationis prioris clarius repraeſententur.

Scholion II.

§. 194. Cum nomen derivativum ſummae angulorum ACB, et D ſit $2\pi - \beta - \lambda - \psi$, et conſequenter $\mathit{ſin}.\,(ACB+D) = -\,\mathit{ſin}\,(\beta+\lambda+\psi)$, nec non $\cos.\,(ACB+D) = \cos.\,(\beta+\lambda+\psi)$, ex his et ex formatis in triangulo ſiniſtro ſinu coſinuque anguli ACB formo ſinum anguli D, quem ut ante reperio:

$$= \frac{\mathit{ſin}.\,(\beta+\psi)\surd(f^2 - b^2\,\mathit{ſin}.\,\lambda^2) - b\,\mathit{ſin}.\,d\,\cos.\,(\beta+\psi)}{f}$$

et conſequenter ad aequationem ſcholii praecedentis pervenio. Similiter ex ſinu coſinuque ſummae angulorum ACB et D, et ex ſinu coſinuque anguli D, formato ſinu et coſinu anguli ACB, ex his denique et ex ſinu coſinuque anguli B, formatur ſinus anguli CAB, qui invenitur eſſe:

$$= \frac{\mathit{ſin}.\,(\beta+\psi)\surd(c^2 - f^2\,\mathit{ſin}.\,\beta^2) - f\,\mathit{ſin}.\,\beta\,\cos.\,(\beta+\psi)}{f}$$

uti

uti in resolutione fuit inventus, excepto quod signum radicale hic sit affirmativum, cum in solutione principali sit negativum, ob signi radicalis ambiguitatem, et consequenter ad priorem aequationem ut ante pervenio.

Coroll. I.

§. 195. Ex aequatione priori statim sequitur esse latus $BC = b =$

$$= \frac{\pm f(\text{fin.}(\beta+\psi)\surd(c^2-f^2\,\text{fin.}\,\beta^2) - f\,\text{fin.}\,\beta\cos.(\beta+\psi))}{c\,\text{fin.}\,\lambda}$$

sed ex aequatione posteriore in Scholio I. inventa statim sequitur esse: $AD = c = f^2\,\text{fin.}\,\beta :$ $(\text{fin.}(\beta+\psi)\surd(f^2-b^2\,\text{fin.}\,\lambda^2) - b\,\text{fin.}\,\lambda\cos.(\beta+\psi))$.

Coroll. II.

§. 196. Ex aequatione primo inventa statim sequitur esse: $\text{fin.}\,\lambda =$

$$= \frac{f(\text{fin.}(\beta+\psi)\surd(c^2-f^2\,\text{fin.}\,\beta^2) - f\,\text{fin.}\,\beta\cos.(\beta+\psi))}{bc}$$

et consequenter ipsum angulum $\lambda =$ *ang. fin.*

$$\frac{f(\text{fin.}(\beta+\psi)\surd(c^2-f^2\,\text{fin.}\,\beta^2) - f\,\text{fin.}\,\beta\cos.(\beta+\psi))}{bc}$$

sed pro angulo ACD invenio esse:

$$\text{tang.}\,\beta = \frac{\text{tang.}\,\psi\surd(f^2-b^2\,\text{fin.}\,\lambda^2) - b\,\text{fin.}\,\lambda}{f\,\text{fec.}\,\psi - \surd(f^2-b^2\,\text{fin.}\,\lambda^2) - b\,\text{fin.}\,\lambda\,\text{tang.}\,\psi}$$

et consequenter ipse angulus

$$\beta = \text{ang.tang.}\frac{(\text{tang.}\,\psi\surd(f^2-b^2\,\text{fin.}\,\lambda^2) - b\,\text{fin.}\,\lambda}{(f\,\text{fec.}\,\psi - \surd(f^2-b^2\,\text{fin.}\,\lambda^2) - b\,\text{fin.}\,\lambda\,\text{tang.}\,\psi}.$$

Coroll.

Coroll. III.

§. 197. Quod ſi aequationes inventae ita combinentur ut prima per $f^2 \mathit{ſin.}\, \beta$, ſecunda per $b\,c\, \mathit{ſin.}\, \lambda$ multiplicetur, ex aequatione ſic prodeunte, pro habendo angulo $A = \psi$, obtinetur haec aequatio: $\mathit{tang.}\ (\beta + \psi) =$

$$= \frac{(f^2 \mathit{ſin.}\, \beta + b\,c\, \mathit{ſin.}\, \lambda)\ (f^2 \mathit{ſin.}\, \beta - b\,c\, \mathit{ſin.}\, \lambda)}{b\,c^2 \mathit{ſin.}\, \lambda \sqrt{(f^2 - b^2 \mathit{ſin.}\, \lambda^2)} - f^3 \mathit{ſin.}\, \beta \sqrt{(c^2 - f^2 \mathit{ſin.}\, \beta^2)}}$$

et conſequenter $\psi = \mathit{ang.\ tang.}$

$$\frac{(f^2 \mathit{ſin.}\, \beta + b\,c\, \mathit{ſin.}\, \lambda)\ (f^2 \mathit{ſin.}\, \beta - b\,c\, \mathit{ſin.}\, \lambda)}{b\,c^2 \mathit{ſin.}\, \lambda \sqrt{(f^2 - b^2 \mathit{ſin.}\, \lambda^2)} - f^3 \mathit{ſin.}\, \beta \sqrt{(c^2 - f^2 \mathit{ſin.}\, \beta^2)}} - \beta.$$

Coroll. IV.

§. 198. Ex utraque aequatione, poſito ad abbreviandum $\mathit{ſin.}\, \beta = m$, $\mathit{ſin.}\, \lambda = h$, $\mathit{ſin.}\, (\beta + \psi) = p$, $\mathit{cos.}(\beta + \psi) = q$, elicitur haec pro diagonali aequatio:

$$f = \frac{\sqrt{(c^2 p^2 - 2\,b\,c\,h\,m\,q \pm c\,p \sqrt{(c^2 p^2 - 4\,b\,h\,m\,(c\,q + b\,h\,m)}}}{m \sqrt{2}}.$$

Scholion III.

§. 199. Omnes caſus hujus problematis, excepto ſexto, etiam Trigonometriae ſubſunt, per quam fere facilius expediuntur, ſed ſolutiones tetragonometricas tamen pro his caſibus elicui, quia ſatis expedite ex aequationibus eruuntur. Sextus caſus quo ex invenienda diagonali figura conſtruenda proponitur, novum in Geometria practica ſiſtit problema, cujus utilitas cum pulchritudine certat, cujus ſolutionem per aequationem quadraticam ſatis utilem et concinnam exhibui.

Coroll.

Coroll. V.

§. 200. Si in aequatione primo inventa ponantur anguli A et ACD ſimul ſumti aequales duobus rectis, evaneſcente termino inter medio fit $f^2 \mathit{ſin}.\beta = bc\,\mathit{ſin}.\lambda$, quae aequatio reſolvitur in hanc analogiam: $f^2 : bc = \mathit{ſin}.\lambda : \mathit{ſin}.\beta$, quae hoc ſuppeditat

Theorema.

Si in quadrilatero angulus combinationis una cum angulo partiali dextro ad diagonalem ſit aequalis duobus rectis; erit quadratum diagonalis ad rectangulum ex lateribus oppoſitis, ut ſinus anguli qui in ſiniſtro triangulo diagonali opponitur, ad ſinum anguli, qui in dextro triangulo, infimo lateri opponitur, ſed idem etiam ex aequatione pro diagonali ſtatim ſequitur cum fiat $f = \frac{\surd(chm}{m}$ et conſequenter $f^2 m = bch$. Si praeterea ponatur $b = c$, erit $f^2 : c^2 = f^2 : b^2 = \mathit{ſin}.\lambda : \mathit{ſin}.\beta$. Si ſit $f = b$; erit $f : c = \mathit{ſin}.\lambda : \mathit{ſin}.\beta$; et ſi $f = c$ erit $f : b = \mathit{ſin}.\lambda : \mathit{ſin}.\beta$.

Coroll. VI.

§. 201. Si iidem anguli ſimul ſumti ſint uni recto aequales, aequatio pro diagonali in hanc mutatur: $f = \frac{\surd(c^2 \pm c\surd(c^2 - 4b^2 \mathit{ſin}.\beta^2 \mathit{ſin}.\lambda^2))}{\mathit{ſin}.\beta \surd 2}$, unde fit, quemadmodum etiam ex aequatione, primo inventa, ſine operatione ſtatim patet, $bc\,\mathit{ſin}.\lambda = f\surd(\surd(c^2 - f^2 \mathit{ſin}.\beta^2)$ vel etiam $bc\,\mathit{ſin}.\lambda = f\surd(c^2 - f^2 \cos.\psi^2)$, quae aequationes in has

has analogias refolvuntur: $f : b\ \mathit{fin}.\ \lambda = c : \sqrt{(c^2 - f^2\ \mathit{fin}.\ \beta^2)} = c : \sqrt{(c^2 - f^2\ \mathit{cos}.\ \psi^2)}$ qua fuppeditat hoc

Theorema.

Si in trapezio angulus combinationis una cum angulo dextro partiali ad diagonalem fuperiori, fit uni recto aequalis, diagonalis erit ad normalem Cb ab extremo fuo fuperiori in finiftrum latus demiffam, ut latus inferius ad continuationem dextri lateris Dd inter fui extremum imum et normalem ab angulo combinationis in fe demiffam.

Coroll. VII.

§. 202. Si ponatur angulus B aequalis duobus rectis, lateribus AB, BC, in diagonalem incidentibus et trapezio in triangulum ACD abeunte; aequatio pro diagonali immutatur in hanc: $c\ \mathit{fin}.(\beta + \psi) = f\ \mathit{fin}.\ \beta$, unde haec analogia: $f : \mathit{fin}.(\beta + \psi) = c : \mathit{fin}.\ \beta$, hoc ipfo igitur aequationes et operationes verificantur. Idem ex aequatione pofteriore in fcholio I. inventa ftatim fequitur; at ex aequatione, quae in prima folutione fuit inventa fequitur fore: $\mathit{tang}.(\beta + \psi) \sqrt{(c^2 - f^2\ \mathit{fin}.\ \beta^2)} = f\ \mathit{fin}.\ \beta$, id quod ex aequatione generali pro diagonali vel altera illa in fcholio I. inventa contra fpem omnem fequi non videtur; quia vero eft $\mathit{fin}.(\beta + \psi)^2 + \mathit{cos}.(\beta + \psi)^2 = 1$, fi aequationis $f^2\ \mathit{fin}.\ \beta^2 = c^2\ \mathit{fin}.(\beta + \psi^2)$ membrum finiftrum per hanc unitatem multiplicetur, et aequatio reducatur, obtinebitur uti prius: $\mathit{tang}.(\beta + \psi) \sqrt{(f^2 - f^2\ \mathit{fin}.\ \beta^2)} = f\ \mathit{fin}.\ \beta$, unde elicitur hoc trigonometricum

Theo-

Theorema.

$f : \surd(c^2 - f^2 \, \text{fin.}\, \beta^2) == \text{tang.}\, (\beta + \psi) : \text{fin.}\, \beta.$
In omni triangulo obliquangulo latus uni obliquo oppositum est ad rectum inter verticem ejusdem anguli, et perpendiculum interceptam, ut tangens ejusdem anguli ad sinum anguli perpendiculo oppositi.

Scholion IV.

§. 203. Existente angulo D quadrilateri $ABCD$ sive trianguli dextri ACD acuto, valet signum radicale affirmativum, et tangentis affirmativum, ob affirmativum cosinum; sed existente angulo D obtuso, valet signum radicale negativum, et tangentis negativum, ob negativum cosinum, ut sit in casu priori: $f : + \surd(c^2 - f^2 \, \text{fin}\, \beta^2) = + \text{tang.}\, (\beta + \psi) : \text{fin.}\, \beta$ at in posteriori, $f : - \surd(c^2 - f^2 \, \text{fin.}\, \beta^2) = - \text{tang.}\, (\beta + \psi) : \text{fin.}\, \beta.$

Coroll. VIII.

§. 204. Quod si ponatur angulus $ACD = \beta = o$, ex aequatione scholii I. manifesto sequitur esse: $\text{tang.}\, \psi \surd(f^2 - b^2 \, \text{fin.}\, \lambda^2) = b \, \text{fin.}\, \lambda.$ At ex aequatione, quae in ipsa resolutione fuit inventa sequitur statim, $b \, \text{fin.}\, \lambda == f \, \text{fin.}\, \psi$, cujus membra si quadrentur, et sinistrum per $\text{fin.}\, \psi^2 + \text{cos.}\, \psi^2$ multiplicetur, facta reductione ex eadem obtinetur etiam $\text{tang.}\, \psi \surd(f^2 - b^2 \text{fin.}\, \lambda^2) = b \, \text{fin.}\, \lambda$, quae aequatio in hanc resolvitur analogiam $b : \surd(f^2 - b^2 \, \text{fin.}\, \lambda^2) == \text{tang.}\, \psi : \text{fin.}\, \lambda$, h. e. $BC : AB = \text{tang.}\, A : \text{fin.}\, B$, ex qua sequitur hoc trigonometricum

Theo-

Theorema.

In omni triangulo obliquangulo, ſi ab angulo uno in latus oppoſitum demittatur normalis; erit latus BC ad ſegmentum AB, ut tangens anguli iſti lateri oppoſiti ad ſinum anguli huic oppoſiti.

Coroll. IX.

§. 205. Si angulus B ſit aequalis tribus rectis, cadente vertice B, ultra diagonalem intra vel extra triangulum ACD, et trapezio inverſo prodeunte, fiet diagonalis:

$$f = \frac{\sqrt{(c^2p^2 + 2bcmq \pm cp\sqrt{(c^2p^2 + 4bm(cq - bm))})}}{m\sqrt{2}}.$$

Si vero angulus A ſit tribus rectis aequalis, cadente vertice A ſupra diagonalem BD, intra vel extra triangulum BCD, et trapezio inverſo rurſus prodeunte, fit pro diagonali haec aequatio:

$$f = \frac{\sqrt{(c^2 \cos.\beta^2 - 2bch \sin.\beta^2 \mp c\cos.\beta}}{\sin.\beta\sqrt{2}}$$
$$\frac{\sqrt{(c^2\cos.\beta^2 - 4bh\sin.\beta^2(c + bh)))}}{\sin.\beta\sqrt{2}}.$$

Coroll. X.

§. 206. Si angulus B ſit rectus ſit aequatio pro diagonali:

$$f = \frac{\sqrt{(c^2p^2 - 2bcmq \pm cp\sqrt{(c^2p^2 - 4bm(cq + bm))})}}{m\sqrt{2}}.$$

Si vero angulus ACD ſit rectus habetur haec:

$$f = \frac{\sqrt{(c^2p^2 - 2bchq \pm cp\sqrt{(c^2p^2 - 4bh(cq + bh))})}}{\sqrt{2}}.$$

Sed si uterque simul rectus ponatur, habetur:

$$f = \frac{\sqrt{(c^2 \cos.\psi^2 + 2bc \sin.\psi} \pm c \cos.\psi}{\sqrt{2}}$$

$$\frac{\sqrt{(c^2 \cos.\psi^2 - 4b(b - c \sin.\psi)))}}{\sqrt{2}}.$$

Si denique angulus A sit rectus, fit aequatio:

$$f = \frac{\sqrt{(c^2 \cos.\beta^2 + 2bch \sin.\beta^2} \pm c \cos.\beta}{\sin.\beta\sqrt{2}}$$

$$\frac{\sqrt{(c^2 \cos.\beta^2 - 4bh \sin.\beta^2 (bh - c)))}}{\sin.\beta\sqrt{2}}.$$

Coroll. XI.

§. 207. Si angulus $ACD = \beta$ negativus ponatur, ut triangulum dextrum intra sinistrum cadat, et prodeat trapezium inversum; aequatio primo inventa in hanc mutatur: $bc \sin.\lambda = -f \sin.(\psi - \beta)\sqrt{(c^2 - f^2 \sin.\beta^2)} + f^2 \sin.\beta \cos.(\psi - \beta)$, at aequatio in scholio primo inventa fit: $f^2 \sin.\beta = bc \sin.\lambda \cos.(\psi - \beta) - c \sin.(\psi - \beta)\sqrt{(f^2 - b^2 \sin.\lambda^2)}$; sed aequatio pro diagonali quoad signa ita mutatur, denotante p $\sin.(\psi - \beta)$. Sed q $\cos.(\psi - \beta)$

$$f = \frac{\sqrt{(c^2p^2 + 2bchmq} \pm cp\sqrt{(c^2p^2 + 4bhm(cq - bhm)))}}{m\sqrt{2}}.$$

Si in his aequationibus ponatur angulus $A = ACD$, fiet: $f^2 \sin.\beta = bc \sin.\lambda$, unde haec analogia: $f^2 : bc = \sin.\lambda : \sin.\beta$ vel $f : b \sin.\lambda = c : f \sin.\beta$ unde haec

Theoremata.

$AC^2 : ADBC = \sin.B : \sin.ACD$ nec non $AC : Cb = AD : Ad$, triangula vero ACb et ADd, sunt similia.

Coroll.

Coroll. XII.

§. 208. Si anguli A et ACD, ponantur ſimul ſumti aequales tribus rectis, ut fiat $p = fin.(\beta+\psi) = -1$, ſed $q = cos.(\beta+\psi) = 0$ fiet

$$f = \frac{\surd(c^2 \pm c \surd(c^2 - 4b^2h^2m^2))}{m\surd 2}$$

Coroll. XIII.

§. 209. Si ponatur $c^2p^2 = 4bhm(cq+bmh)$ ut ſignum radicale dextimum evaneſcat, fiet aequatio pro diagonali $f = \frac{\surd(c^2p^2 - 2bchmq)}{m\surd 2}$ vel etiam pro c^2p^2 ſubſtituto ſuo valore

$$f = \frac{\surd(bh(cq+2bmh)}{\surd m} =$$

$$= \frac{\surd b\, fin.\lambda\,(c\,cos.(\beta+\psi) + 2b\, fin.\lambda\, fin.\beta)}{\surd fin.\beta}$$

Fit autem in hoc caſu per hanc poſitionem: latus $AD = c = \frac{2bhm(q \pm 1)}{p^2}$, quod ſi vero ponatur $c^2p^2 = 2bchmq$ fiet

$$f = \frac{\surd(cp\surd(-2bhm(cq+bhm)}{m\surd 2},$$ quae aequatio eſt imaginaria niſi vel h, vel m, vel q, negative ſumatur, ſic enim fiet

$$f = \frac{\surd(cp\surd(2bhm(cq-bhm)))}{m\surd 2},$$ fit autem in hoc caſu $c = \frac{2bhmq}{p^2} = \frac{2b\, fin.\lambda\, fin.\beta\, cos.(\beta+\psi)}{fin.(\beta+\psi)^2}$, inde ſequitur, $c\; tang.(\beta+\psi) = fin.\lambda\; fin.\beta : fin.(\beta+\psi)$.

Coroll. XIV.

§. 210. Si in aequatione primo inventa ponatur $c=f$, et adhibeatur ſignum radicale negativum ut in reſolutione prodiit; aequatio mutatur in hanc: $b\,\text{ſin.}\,\lambda = -f\,\text{ſin.}\,(2\beta+\psi)$, unde haec analogia $b : -\text{ſin.}\,\text{ſin.}\,(2\beta+\psi) = f : \text{ſin.}\,\lambda$. Si vero in aequatione in Scholio I. inventa ponatur $b=f$ et ſignum affirmativum adhibeatur; aequatio mutatur in hanc: $f\,\text{ſin.}\,\beta = c\,\text{ſin.}(\beta+\psi-\lambda)$, unde haec analogia: $f : \text{ſin.}\,(\beta+\psi-\lambda) = c : \text{ſin.}\,\beta$, quarum analogiarum utraque vulgaria Theoremata trigonometrica ſiſtit, et conſequenter operationes verificat.

Scholion V.

§. 211. Aequatio inventa in reſolutione hujus problematis pro trapezio directo valet, excedente ſumma angulorum ACB et D angulum rectum, verum exiſtente eadem recto minore et ſimul angulo A majore angulo ACD; valet haec aequatio: $b\,c\,\text{ſin.}\,\lambda = f\,\text{ſin.}\,(\psi-\beta)\,\sqrt{}(c^2-f^2\,\text{ſin.}\,\beta^2-f^2\,\text{ſin.}\,\beta\,\cos.(\psi-\beta)$. Sed exiſtente angulo ACD majore angulo A valet haec: $b\,c\,\text{ſin.}\,\lambda = -f\,\text{ſin.}(\beta-\psi)\sqrt{}(c^2-f^2\,\text{ſin.}\,\beta^2)-f^2\,\text{ſin.}\,\beta\,\cos.\,(\beta-\psi)$. Generalis igitur aequatio pro omni trapezio directo eſt talis: $b\,c\,\text{ſin.}\,\lambda = \mp f\,\text{ſin.}\binom{\beta\pm\psi}{\psi-\beta}\sqrt{}(c^2-f^2\,\text{ſin.}\,\beta^2)-f^2\,\text{ſin.}\,\beta\,\cos.\binom{\beta\pm\psi}{\psi-\beta}$.

Praeter has variationes non occurrunt niſi quae ex ambiguitate ſigni radicalis et coſinus oriuntur. Si trapezium ſit partialiter inverſum, ut totum triangulum ADC cadat intra triangulum ABC, exiſtente ſumma angulorum ACD et D recto

majore,

majore, et angulo A majore angulo ACD; aequatio prima cum ſignis contrariis reproducitur; ſed exiſtente angulo ACD maiore angulo A; reproducitur aequatio ſecunda cum ſignis contrariis. Quod ſi vero ſumma angulorum D et ACD ſit minor recto; reproducitur aequatio, quae in ipſa ſolutione problematis fuit inventa cum ſignis contrariis. Caeterum quomodocunque trapezium ſit ſive partialiter ſive totaliter inverſum; ex ſimplicitate et forma aequationis patet, plures jam allatis variationes in ipſa occurrere non poſſe, quare generaliſſima aequatio erit hujusmodi: $bc\,\text{ſin.}\,\lambda = \mp f\,\text{ſin.}\begin{pmatrix}\beta\pm\psi\\ \psi-\beta\end{pmatrix}$
$\sqrt{(c^2-f^2\,\text{ſin.}\,\beta^2)} = f^2\,\text{ſin.}\,\beta\,\text{cos.}\begin{pmatrix}\beta\pm\psi\\ \psi-\beta\end{pmatrix}$.

Problema XV.

§. 212. In figura quadrilatera $ABCD$, inter haec ſex: latus $AB=a$, $BC=b$, $AD=c$, diagonalem $AC=f$, angulum $A=\psi$, $ACD=\beta$: aequationem invenire. **Fig. XV.**

Solutio.

1). Cum ſit ex reſolutione problematis XII. (§. 165. num. 1.) $Ab = \frac{a^2+f^2-b^2}{2\psi}$ vel $\frac{b^2+f^2-a^2}{2f}$ et ex reſolutione problematis antecedentis (§. 192. num. 2.) habeatur:

$$\text{ſin.}\,CAD = \frac{\text{ſin.}\,\beta\sqrt{(c^2-f^2\,\text{ſin.}\,\beta^2)}+f\,\text{ſin.}\,\beta\,\text{cos.}\,\beta}{c},$$

$$\text{et cos.}\,CAD = \frac{-\text{cos.}\,\beta\sqrt{(c^2-f^2\,\text{ſin.}\,\beta^2)}+f\,\text{ſin.}\,\beta^2}{c};$$

 ex

ex his et ex finu cofinuque anguli A formo cofinum anguli CAB, quem invenio effe $= cos. \psi - CAD =$

$$= \frac{-cos.\psi\, cos.\beta \sqrt{(c^2 - f^2 fin.\beta^2)} + f\, cos.\psi\, fin.\beta^2}{c}$$

$$\frac{+ fin.\psi\, fin.\beta \sqrt{(c^2 - f^2 fin.\beta^2)} + f\, fin.\psi\, fin.\beta\, cos.\beta}{c}$$

$$= \frac{-cos.(\beta+\psi) \sqrt{(c^2 - f^2 fin.\beta^2)} + f\, fin.\beta\, fin.(\beta+\psi)}{c}$$

2) Hinc in triangulo rectangulo ABb ad hanc analogiam pervenio $1 : AB = cos.\, CAB : AB$ h. e.

$$1 : a = \frac{-\, cos.(\beta+\psi) \sqrt{(c^2 - f^2 fin.\beta^2)} +}{c}$$

$$\frac{f\, fin.\beta\, fin.(\beta+\psi)}{c} : \frac{a^2 + f^2 - b^2}{2f}$$ et hanc

aequationem obtineo: $\pm (a^2 + f^2 - b^2)\, c = 2af^2 fin.\beta$ $fin.(\beta+\psi) - 2af\, cos.(\beta+\psi) \sqrt{(c^2 - f^2 fin.\beta^2)}$ ubi, fi latus $AB > BC$, valet fignum fuperius, fed fi $BC > AB$, inferius. Q. E. I.

Coroll. I.

§. 213. Ex hac aequatione ftatim obtinetur latus $BC = b = \frac{\sqrt{((a^2 \pm f^2)\, c \pm 2af\, cos.(\beta+\psi)}}{\sqrt{c}}$

$$\frac{\sqrt{(c^2 - f^2 fin.\beta^2)} \mp 2af^2 fin.\beta\, fin.(\beta+\psi))}{\sqrt{c}}$$

Ubi figna fuperiora valent, fi latus $AB > BC$, fed inferiora, fi latus $AB < BC$. Latus vero AB obtinetur per hanc aequationem quadraticam ad-

adfectam: $a = \frac{\pm f(fmp - q\sqrt{(c^2 - f^2 m^2)})}{c}$

$\frac{\pm \sqrt{((fmp - q\sqrt{((c^2 - f^2 m^2)^2 f^2 + (b^2 \pm f^2) c^2)}}}{c}$

Ubi si latus $BC > AB$ valet signum negativum termini finiſtimi in dextro aequationis membro, ſed affirmativum quadrati f^2 ſub ſigno radicali. Sed in caſu quo $AB > BC$ valet affirmativum in termino finiſtimo dextri membri, ſed negativum quadrati f^2 ſub ſigno radicali; poſui autem ad abbreviandum $\mathit{ſin.}\ \beta = m$, $\mathit{ſin.}\ (\beta + \psi) = p$, et $\mathit{cos.}\ (\beta + \psi) = q$.

Coroll. II.

§. 214. Sit ad abbreviandum $\frac{a^2 + f^2 - b^2}{2af}$ vel etiam $\frac{b^2 + f^2 - a^2}{2af} = h$, ſed ſinibus impoſita nomina maneant, et factis ſubſtitutionibus erit aequatio abbreviata: $q\sqrt{(c^2 - f^2 m^2)} = fmp - bc$, ex qua operatione abſoluta invenitur eſſe: latus $AD = c = \frac{fm(-hp \pm q\sqrt{(1 - h^2)})}{q^2 - h^2}$

Coroll. III.

§. 215. Pro invenienda diagonali, manentibus ad abbreviandum iisdem nominibus, factis ſubſtitutionibus et transpoſitionibus, erit aequatio abbreviata: $f^2(2amp - c) \mp (a^2 - b^2)c = 2afq\sqrt{(c^2 - f^2 m^2)}$, ex qua ſumtis quadratis, facta reductione, concinnatione et omni operatione

K 4 abſo-

absoluta habetur haec aequatio quadratica affecta:

$$f=\frac{\sqrt{(2a^2c^2q^2-(a^2-b^2)(c^2-2acmp)}\pm 2acq}{\sqrt{(4am(am-cp)+c^2)}}$$

$$\frac{\sqrt{(a^2c^2q^2-(a^2-b^2)(c^2-2acmp)-m^2(a^2-b^2)))}}{\sqrt{(4am(am-cp)+c^2)}}$$

Scholion I.

§. 216. Cum ex resolutione problematis XII. n. 3. sit $fin. D=\frac{fin.(\beta+\psi)\sqrt{(a^2-h^2)}-h\,cos.(\beta+\psi)}{a}$, ubi positum fuit $h=\frac{\sqrt{(a^2-(a^2+f^2-b^2)^2)}}{4f^2}$ in triangulo dextro, statim habetur haec analogia $AC: fin. D == AD: fin. ACD$, h. e. $f:\frac{fin.(\beta+\psi)\sqrt{(a^2-h^2)}-h\,cos.(\beta+\psi)}{a}=c: fin. \beta$ et consequenter haec aequatio: $af\,fin.\beta == c\,fin.(\beta+\psi)\sqrt{(a^2-h^2)}-ch\,cos.(\beta+\psi)$.

Coroll. IV.

§. 217. Ex hac aequatione obtinetur pro angulo ACD obtinendo haec aequatio:

$$tang.\beta=\frac{c(fin.\psi)\sqrt{(a^2-h^2)}-h\,cos.\psi)}{af-c(h\,fin.\psi+cos.\psi\sqrt{(a^2-h^2)})},$$ vel etiam

$$tang.\beta=\frac{c\,tang.\psi\sqrt{(a^2-h^2)}-ch,}{af\,fec.\psi-ch\,tang.\psi-c\sqrt{(a^2-h^2)}},$$

et consequenter

$$\beta=ang.tang.\frac{(c\,tang.\psi\sqrt{(a^2-h^2)}-ch}{(af\,fec.\psi-ch\,tang.\psi-c\sqrt{(a^2-h^2)})},$$

ex

ex eadem aequatione pro angulo A pervenio ad hanc aequationem:

$$\text{fin.}\,(\beta+\psi)=\frac{f\,\text{fin.}\,\beta\,\sqrt{(a^2-h^2)}\pm h\sqrt{(c^2-f^2\,\text{fin.}\,\beta^2)}}{a\,c}$$

et hinc ipfe angulus:

$$\psi = \text{ang. fin.}\frac{(f\,\text{fin.}\,\beta\,\sqrt{(a^2-h^2)}\pm h\sqrt{(c^2-f^2\,\text{fin.}\,\beta^2)})}{a\,c}-\beta.$$

Scholion II.

§. 218. Solus cafus, quo diagonalis quaeritur Tetragonometriae proprius, novum in Geometria practica fuppeditat problema fatis utile, pulchrum et elegans. Caeteri cafus quinque utrique methodo et trigonometricae et tetragonometricae fubfunt, fed trigonometrice facilius expediuntur, quos tamen evolvi cum ex aequatione breviter et expedite fequantur.

Coroll. V.

§. 219. Si anguli A et ACD fimul fumti fint aequales duobus rectis, aequatio pro diagonali in hanc contrahitur:

$$f=\frac{\sqrt{\left(2a^2c^2-(a^2-b^2)\,c^2\pm 2ac\right.}}{\sqrt{(c^2+4a^2m^2)}}\;\frac{\left.(\sqrt{a^2c^2-(a^2-b^2)\,(c^2+m^2(a^2-b^2))})\right)}{\sqrt{(c^2+4a^2m^2)}}$$

Quod fi vero iidem anguli fimul fumti fint uni recto aequales; aequatio in hanc contrahitur: $f=\frac{\sqrt{(a^2-b^2)'c}}{2am-c}$. Sed in hypothefi angulorum

 A

A et ACD duobus rectis aequalium, sequitur ex aequatione in scholio I. inventa esse $f\ \text{sin.}\ \beta = \frac{ch}{a}$ et consequenter $c : \text{sin.}\ \beta = : \frac{h}{a}$, quam analogiam veram esse patet ex eo, quod ex hypothesi sequatur angulum CAB, cujus sinus est $\frac{h}{a}$, aequalem esse angulo D. Sed in hypothesi eorundem uni recto aequalium fit ex eadem aequatione: $f\ \text{sin.}\ \beta = \frac{c\sqrt{(a^2 - h^2)}}{a}$ unde haec analogia: $c : \text{sin.}\ \beta = f : \frac{\sqrt{(a^2 - h^2)}}{a}$, quam si in hypothesin et figuram oculos et mentis aciem dirigas, veram esse videbis.

Coroll. VI.

§. 220. Quod si angulum ACD evanescere ponamus, lateribus AD, DC, in diagonalem AC incidentibus et trapezio in triangulum ABC abeunte; aequatio pro diagonali in hanc abit: $f = \sqrt{(2a^2q^2 + b^2 - a^2 \pm 2aq\sqrt{(a^2q^2 + b^2 - a^2)})}$, et hinc rursus extracta radix suppeditat hanc aequationem: $f = \pm aq \pm \sqrt{(a^2q^2 - a^2 + b^2)}$, vel etiam $f = \pm a\ \cos.\psi \pm \sqrt{(a^2 \cos.\psi^2 - a^2 + b^2)}$, quae in hanc mutatur: $f = \pm a\ \cos.\psi \pm \sqrt{(b^2 - a^2\ \text{sin.}\ \psi^2)}$; id quod per elementa Geometriae manifeste verum esse patet, et operationes consequenter hoc ipso verificantur.

Coroll. VII.

§. 221. Si anguli A et ACD ponantur ſimul aequales tribus rectis; aequatio generalis in hanc abit: $f = \frac{\sqrt{(b^2 - a^2)c}}{\sqrt{(c + 2am)}} = \frac{\sqrt{(b^2 - a^2)c}}{(c + 2a \, \text{ſin.} \, \beta)}$, quae ne ſit imaginaria, debet eſſe latus BC majus latere AC, vel ſi minus, debet eſſe angulus ACD negativus et $c < 2a \, \text{ſin.} \, \beta$. Sed ex aequatione ſcholii I. fit $-f \, \text{ſin.} \, \beta = \frac{-c\sqrt{(a^2 - h^2)}}{a}$, unde haec analogia $c : -\text{ſin.} \, \beta = f : \frac{-\sqrt{(a^2 - h^2)}}{a}$.

Coroll. VIII.

§. 222. Quod ſi latera AD, et BC ſint aequalia inter ſe; aequatio pro diagonali in hanc contrahitur: $f = 2acq : \sqrt{(4am(am - cp) + c^2)}$, ſi vero ponatur $c = 2amp$, aequatio in hanc contrahitur:

$$f = \frac{\sqrt{(2a^2c^2q^2 \pm 2acq\sqrt{(a^2c^2q^2 + m^2(a^2 - b^2)^2)})}}{2amq},$$

vel etiam $f = \frac{\sqrt{(c(acq \pm \sqrt{(a^2c^2q^2 + m^2(a^2 - b^2)^2)}))}}{m\sqrt{2aq}}$.

Coroll. IX.

§. 223. Si angulus A ponatur aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus, et trapezio in triangulum BCD abeunte, pro reſolvendo novo problemate ad triangula pertinente, haec prodit

dit aequatio pro recta AC, ab angulo C in latus oppositum BD ducta.

$$f = \frac{r\,(2a^2c^2q^2 - (a^2-b^2)\,(c^2+2acm^2) \pm 2acq}{r\,(4am^2(a+c^2)+c^2)}$$
$$\frac{r\,(a^2c^2q^2-(a^2-b^2)\times(c^2+2acm^2)-m^2(a^2-b^2)))}{r\,(4am^2(a+c^2)+c^2)}$$

Scholion III.

§. 224. Sed in hac aequatione littera q denotat $-\cos.\beta$, cum in aequatione generali denotet $\cos.(\beta+\psi)$, verum in triangulo totali non dantur nisi duo latera BD, BC, et pars ECD anguli C, in partiali triangulo EBC non dantur nisi duo latera BE, BC, in partiali DCE datur latus AD et angulus oppositus ECD.

Coroll. X.

§. 225. Si in aequatione primo inventa ponatur $c=f$, aequatio adhibito signo radicali affirmativo mutatur in hanc $a^2+f^2-b^2=-2af$ $\cos.(2\beta+\psi)$, unde fit $b^2=a^2+f^2+2af$ $\cos.(2\beta+\psi)$, id quod per inspectionem figurae et principia geometrica verum esse liquet, quo igitur aequatio verificatur.

Scholion IV.

§. 226. Aequatio hujus problematis pro trapezio directo valet in hypothesi figurae constructae, cadente perpendiculo intra figuram et summa angulorum D et ACD excedente rectum; verum existente eadem recto minore et angulo ACD angulum A excedente, valet haec aequa-

aequatio: $(a^2+f^2-b^2)\,c = 2af\cos.(\beta-\psi)$ $\surd(c^2-f^2\sin.\beta^2) - 2af^2\sin.\beta\sin.(\beta-\psi)$ vel etiam haec altera: $(a^2+f^2-b^2)\,c = 2af\cos.(\psi-\beta)$ $\surd(c^2-f^2\sin.\beta^2)+2af^2\sin.\beta\sin.(\psi-\beta)$, quare generalis aequatio pro omni trapezio directo erit talis:

$$\begin{pmatrix}a^2+f^2-b^2\\ b^2+f^2-a\end{pmatrix} c = \mp 2af\cos.\begin{pmatrix}\beta\pm\psi\\ \psi-\beta\end{pmatrix}$$
$$\surd(c^2-f^2\sin.\beta^2)\pm 2af^2\sin.\beta\sin.\begin{pmatrix}\beta\pm\psi\\ \psi-\beta\end{pmatrix}$$

Si trapezium ita ſit partialiter inverſum ut totum triangulum *ADC* cadat intra triangulum *ABC*, excedente ſumma angulorum *D* et *ACD* rectum, et angulo hoc illo majore, reproducitur aequatio prima cum ſignis contrariis; ſed angulo *A* excedente angulum *ACD* recurrit ſecunda cum ſignis contrariis. Verum exiſtente ſumma eorundem recto minore; recurrit ipſa aequatio in ſolutione problematis inventa cum ſignis contrariis. Quod ſi trapezium ita ſit totaliter inverſum, ut latus *AD* ſit intra latus *AB*, latus autem *CD* extra latus *CB*, eaedem variationes reproducentur, verum quomodocunque figura varietur, ex forma aequationis patet plures variationes locum habere non poſſe, quare generalis aequatio allata pro trapezio directo, valet etiam pro omni inverſo.

Problema XVI.

§. 227. In figura quadrilatera propoſita *ABCD*, inter haec ſex: latus $AB=a$, $AD=c$, diagonalem $AC=f$, et tres angulos $A=\psi$, $B=\lambda$ et denique $ACD=\beta$; aequationem invenire. **Fig. XVI. 1. 2.**

Solutio.

Solutio.

Cum fit in ploblem. XIII. fchol. I. (§. 176.) inventus $fin. D = -\frac{-fin.(\beta+\lambda+\psi)\,\Gamma(f^2-a^2 fin.\lambda^2)}{f}$ $\frac{-a fin.\lambda\, cos.(\beta+\lambda+\psi)}{f}$ in triangulo dextro flatim pervenio ad hanc analogiam: $AC : fin. D = AD : fin. ACD$, h. e. $f : \frac{-fin.(\beta+\lambda+\psi)\,\Gamma(f^2-a^2 fin.\lambda^2)}{f}$ $\frac{-a fin.\lambda\, cos.(\beta+\lambda+\psi)}{f} = c : fin.\beta$ et confequenter habetur haec aequatio: $f^2 fin.\beta = - c$ $fin.(\beta+\lambda+\psi)\,\Gamma(f^2 - a^2 fin.\lambda^2) - ac\, fin.\lambda$ $cos.(\beta+\lambda+\psi)$.

Scholion I.

§. 228. Ad eandem aequationem paullo aliter pervenitur hoc modo: Cum fit finus fummae angulorum A, B, ACD, $fin.(\beta+\lambda+\psi)$ et cofinus $cos.(\beta+d+\psi)$, finus autem anguli ACB fit $\frac{a\, fin.\lambda}{f}$ et cofinus, $\frac{\Gamma(f^2 - a^2 fin. d^2)}{f}$; erit ex his formatus finus fummae omnium angulorum $A, B, C, = \frac{fin.(\beta+\lambda+\psi)\,\Gamma(f^2-a^2 fin.\lambda^2)}{f}$ $\frac{+a fin.\lambda\, cos.(\beta+d+\psi)}{f} = fin.\ (A+B+C)$. Quare cum fit propter fummam omnium angulorum in quadrilatero aequalem quatuor rectis; angulus $D = 2\pi - A - B - C$; erit; $fin. 2\pi - (A+B+C)$ $=$

$$= - \text{fin.}(A+B+C) = \frac{-\text{fin.}(\beta+\lambda+\psi)}{f}$$
$$\frac{\sqrt{(f^2 - a^2 \text{fin.}\lambda^2)} - a\,\text{fin.}\lambda\,\text{cos.}(\beta+\lambda+\psi)}{f}$$ ob fin $2\pi = o$ et $\text{cos.}\ 2\pi = 1$, et confequenter ad eandem ac prius aequationem pervenitur.

Solutio altera.

1) Cum refolutione probl. XIV. (§. 192.) habeatur:
$$\text{fin.}\,CAD = \frac{\text{fin.}\beta\sqrt{(c^2 - f^2\text{fin.}\beta^2)} + f\,\text{fin.}\beta\,\text{cos.}\beta}{c}$$ et
$$\text{cos.}\,CAD = \frac{-\text{cos.}\beta\sqrt{(c^2 - f^2\text{fin.}\beta^2)} + f\,\text{fin.}\beta^2}{c}$$
et ex his in folutione problematis XV. (§. 212. num. 1.) formatus fit
$$\text{cos.}\ CAB = \frac{-\text{cos.}(\beta+\psi)\sqrt{(c^2 - f^2\text{fin.}\beta^2)} + f\,\text{fin.}\beta\,\text{fin.}(\beta+\psi)}{c}$$
reftat formatio finus ejusdem anguli, et obtinetur:
$$\text{fin.}\ (\psi - CAD) = \frac{-\text{fin.}\psi\,\text{cos.}\beta\sqrt{(c^2 - f^2\text{fin.}\beta^2)} + f\,\text{fin.}\beta^2\,\text{fin.}\psi}{c}$$
$$\frac{-\text{fin.}\beta\,\text{cos.}\psi\sqrt{(c^2 - f\,\text{fin.}\beta^2)} - f\,\text{fin.}\beta\,\text{cos.}\beta\,\text{cos.}\psi}{c}$$
$$= \frac{-\text{fin.}(\beta+\psi)\sqrt{(c^2 - f^2\text{fin.}\beta^2)} - f\,\text{fin.}\beta\,\text{cos.}(\beta+\psi)}{c}$$
$$= \text{fin.}\ CAB.$$

2) Ex his et ex finu cofinuque anguli B formo finum anguli tertii ACB, quem invenio effe

=

$$= \frac{-\sin.\lambda\cos.(\beta+\psi)\sqrt{(c^2-f^2\sin.\beta^2)}}{c}$$
$$+ \frac{f\sin.\beta\sin.\lambda\sin.(\beta+\psi)}{c}$$
$$- \frac{\cos.\lambda\sin.(\beta+\psi)\sqrt{(c^2-f^2\sin.\beta^2)}}{c}$$
$$- \frac{f\sin.\beta\cos.\lambda\cos.(\beta+\psi)}{c}$$
$$= \frac{-\sin.(\beta+\lambda+\psi)\sqrt{(c^2-f^2\sin.\beta^2)}}{c}$$
$$\frac{-f\sin.\beta\cos.(\beta+\lambda+\psi)}{c} = \sin.ACB.$$

3) Jam in triangulo finistro pervenio ad hanc analogiam: $AC : \sin. B = AB : \sin. ACB$, h. e. $f : \sin.\lambda = a : \frac{-\sin.(\beta+\lambda+\psi)\sqrt{(c^2-f^2\sin.\beta^2)}}{c}$ $\frac{-f\sin.\beta(\cos.(\beta+\lambda+\psi)}{c}$ et consequenter obtinetur haec aequatio: $ac\sin.\lambda = -f\sin.(\beta+\lambda+\psi)$ $\sqrt{(c^2-f^2\sin.\beta^2)} - f^2\sin.\beta\cos.(\beta+d+\psi)$. Q. E I.

Scholion II.

§. 229. Formatio finus anguli ACB etiam aliter obiineri poterat et quidem hoc modo. Sinus fummae angulorum ACB et D, eft $-\sin.(\beta+\lambda+\psi)$ et cofinus $\cos.(\beta+d+\psi)$, finus autem anguli D eft $\frac{f\sin.\beta}{c}$ et cofinus $\sqrt{(c^2-f^2\sin.\beta^2)}$ hinc erit:

fin.

$$fin.\,(A\,C\,B+D-D)=\frac{-\,fin.\,(\beta+d+\psi)}{c}$$

$$\frac{\sqrt{(c^2-f^2 fin\,\beta^2)}-f\,fin.\,\beta\,cos.(\beta+\lambda+\psi)}{c}=fin.\,ACB,$$

Idem prorſus ac ante, et conſequenter ad eandem aequationem pervenio:

Scholion III.

§. 230. Prior aequatio eſt poſterioris, et poſterior prioris radix quadratica negativa; nam ſi ex hac poſteriore quaeras valorem $f^2\,fin.\,\beta$, obtinebis $f^2\,fin.\,\beta =$ $= -a\,c\,fin.\,\lambda\,cos.\,(\beta+\lambda+\psi) \pm c\,fin.\,(\beta+\lambda+\psi)$ $\sqrt{(f^2-a^2\,fin.\,d^2)}$ et ſi ex priore quaeras valorem $a\,c\,fin.\,\lambda$, habebis $a\,c\,fin.\,\lambda = -f^2 fin.\,\beta\,cos.(\beta+\lambda+\psi)$ $\pm f\,fin.+d+\psi)\sqrt{(c^2-f^2\,fin.\,\beta^2)}$, quamobrem etiam ob ambiguitatem ſigni radicalis, terminum dextri membri ſigno radicali affectum ponere licet affirmativum, ut radices affirmativae prodeant; aequationes autem non realiter, ſed formaliter, differunt.

Coroll. I.

§. 231. Ex aequatione priore ſtatim obtinetur latus $AD = c = f^2\,fin.\,\beta : (c\,fin.\,(\beta+\lambda+\psi)$ $\sqrt{(f^2-a^2 fin.\,\lambda^2)} - a\,c\,fin.\,\lambda\,cos.\,(\beta+\lambda+\psi))$. Sed ex aequatione poſteriore ſtatim incurrit in oculos, eſſe latus $A\,B = a =$

$$\frac{f\,fin.(\beta+d+\psi)\sqrt{(c^2-f^2 fin.\,\beta^2)}-f^2 fin\,\beta\,cos.(\beta+\lambda+\psi)}{c\,fin.\,\lambda.}$$

Coroll. II.

§. 232. Pro angulo ACD obtinendo habetur ex aequatione priore haec tangentialis aequatio:

$$\text{tang.}\ \beta = \frac{c\,\text{tang.}(\lambda+\psi)\sqrt{(f^2-a^2\,\text{ſin.}\,\lambda^2)}-ac\,\text{ſin.}\,\lambda}{f^2\,\text{ſec.}(\lambda+\psi)-ac\,\text{ſin.}\,\lambda\,\text{tang.}(\lambda+\psi)-c\sqrt{(f^2-a^2\,\text{ſin.}\,\lambda)}},$$

Sed ex aequatione poſteriore habetur pro angulo B

$$\text{tang.}\ \lambda = \frac{f\,\text{tang.}(\beta+\psi)\sqrt{(c^2-f^2\,\text{ſin.}\,\beta^2)}-f^2\,\text{ſin.}\,\beta}{-ac\,\text{ſec.}(\beta+\psi)-f^2\,\text{ſin}\,\beta\,\text{tang.}(\beta+\psi)-f\sqrt{(c^2-f^2\,\text{ſin.}\,\beta^2)}}$$

conſequenter ipſi anguli:

$$\beta = \text{ang. tang.}\ \frac{(c\,\text{tang.}(\lambda+\psi)\sqrt{(f^2-a^2\,\text{ſin.}\,\lambda^2)}-ac\,\text{ſin.}\,\lambda}{(f^2\,\text{ſec.}(\lambda+\psi)-ac\,\text{ſin.}\,\lambda\,\text{tang.}(\lambda+\psi)-c\sqrt{(f^2-a^2\,\text{ſin.}\,\lambda^2)})}$$

et

$$\lambda = \text{ang. tang.}\ \frac{f\,\text{tang.}(\beta+\psi)\sqrt{(c^2-f^2\,\text{ſin.}\,\beta^2}-f^2\,\text{ſin.}\,\beta}{ac\,\text{ſec.}(\beta+\psi)-f^2\,\text{ſin.}\,\beta\,\text{tang.}(\beta+\psi)-f\sqrt{(c^2-f^2\,\text{ſin.}\,\beta^2)}}$$

Coroll. III.

§. 233. Pro obtinendo angulo A combinentur aequationes ita, ut prior per $ac\,\text{ſin.}\,\lambda$, poſterior per $f^2\,\text{ſin.}\,\beta$ multiplicetur, et ex aequatione ſic prodeunte obtinebitur haec aequatio tangentialis:

$$\text{tang.}\ (\beta+\lambda+\psi) = \frac{(ac\,\text{ſin.}\,\lambda+f^2\,\text{ſin.}\,\beta\sqrt{(ac\,\text{ſin.}\,\lambda-f^2\,\text{ſin.}\,\beta}}{ac^2\,\text{ſin.}\,\lambda\sqrt{(f^2-a^2\,\text{ſin.}\,\lambda^2)}-f^3\,\text{ſin.}\,\beta\sqrt{(c^2-f^2\,\text{ſin.}\,\beta^2)}}$$

et conſequenter ipſe angulus quaeſitus erit:

$$\psi = \text{ang. tang.}\ \frac{(ac\,\text{ſin.}\,\lambda+f^2\,\text{ſin.}\,\beta)\,(ac\,\text{ſin.}\,\lambda-f^2\,\text{ſin.}\,\beta)-\beta-\lambda}{ac^2\,\text{ſin.}\,\lambda\sqrt{(f^2-a^2\,\text{ſin.}\,\lambda^2)}-f^3\,\text{ſin.}\,\beta\sqrt{(c^2-f^2\,\text{ſin.}\,\beta^2)}}$$

Coroll.

Coroll. IV.

§. 234. Posito ad abbreviandum $fin. \beta = m$, $fin. \lambda = h$, $fin. (\beta + d + \psi) = p$, $cos. (\beta + d + \psi) = q$, pro diagonali ex utraque aequatione idem hic valor elicitur: $f = \frac{\sqrt{(c^2p^2 - 2acmhq \pm cp}}{m\sqrt{2}}$
$\frac{\sqrt{(c^2p^2 - 4amh(cq + amh)))}}{m\sqrt{2}}$

Scholion IV.

§. 235. Inter fex cafus in aequatione contentos et jam folutos, non datur nifi unus Tetragonometriae proprius, quo diagonalis quaeritur, qui igitur problema novum utile juxta ac pulchrum in Geometria practica fuppeditat. Caeteri quinque cafus, utrique methodo et trigonometricae et tetragonometricae fubfunt, ita tamen ut trigonometrice expeditius folvantur, quos tamen ideo evolvi, quia ex aequationibus facile deducuntur.

Coroll. V.

§. 236. Si omnes anguli A, B, A, C, D, fimul fumti aequentur duobus rectis, aequatio pro diagonali in hanc mutatur $f = \frac{\sqrt{ach}}{m}$, unde fit $f^2 fin. \beta = ac fin. \lambda$, quod idem ex aequationibus primo inventis flatim confequitur, quae aequatio in hanc refolvitur analogiam: $f^2 : ac = fin. \lambda : fin. \beta$ unde hoc

Theorema.

Si in trapezio tres anguli A, B, ACD, fimul fumti aequentur duobus rectis; erit quadratum

 diago-

diagonalis ad rectangulum ex lateribus contiguis AB, AD, ut finus anguli, qui diagonali opponitur in triangulo finiftro, ad finum anguli adjacentis in dextro. Poteft etiam refolvi in hanc analogiam $f : a\, fin.\, d = c : f\, fin.\, \beta$, unde hoc

Theorema.

Diagonalis eft ad normalem Ab in latus BC demiffam, ut latus AD ad normalem Ad, in latus CD demiffam, et triangulum ACb eft fimile triangulo ADd.

Coroll. VI.

§. 237. Si omnes anguli A, B, ACD fimul fumti fint uni recto aequales; aequatio generalis pro diagonali hoc accipit compendium $f = \frac{\sqrt{(c^2 \pm c \sqrt{(c^2 - 4a^2 fin.\, \lambda^2 fin.\, \beta^2)})}}{fin.\, \beta \sqrt{2}}$, fed ex aequatione, ut in refolutione fuit inventa, habetur: $f^2 fin.\, \beta = -c \sqrt{(f^2 - a^2 fin.\, \lambda^2)}$ unde haec analogia: $f : -\sqrt{(f^2 - a^2 fin.\, \lambda^2)} = c : f\, fin.\, \beta$, et hinc fequens

Theorema.

Si tres anguli A, B, ACD, fint fimul fumti uni recto aequales; erit diagonalis ad rectam Cc, inter angulum C, et perpendiculum Ac interceptam, ut latus AD, ad perpendiculum, in latus CD demiffum. Sed ex aequatione pofteriore fequitur: $ac\, fin.\, \lambda = f \sqrt{(c^2 - f^2 fin.\, \beta^2)}$, unde haec analogia: $f : a\, fin.\, \lambda = c : \sqrt{(c^2 - f^2 fin.\, \beta^2)}$, et hoc

Theorema.

In eadem hypothefi eft diagonalis ad perpendiculum Ab; ut latus AD ad rectam Dc; unde fequi-

ſequitur triangula ACc et ADa eſſe ſimilia. Sed poſitis iisdem angulis ſimul ſumtis tribus rectis aequalibus; pro diagonali eadem prodit aequatio, quae ſi reducatur ad formam aequationis primo inventae, vel etiam in eandem haec poſitio immediate inferatur; prodit $f^2 \, \mathit{ſin}. \beta = + c \sqrt{(f^2 - a^2 \, \mathit{ſin}. \lambda^2)}$, unde eadem analogia, idemque priori ſequitur theorema: Sed in hoc caſu ſignum radicale affirmativum indicat, perpendiculum ab angulo A in latus BC demiſſum cadere ſemper verſus ſiniſtram a diagonali; cum in priori poſitione ſemper cadat ad dextram, id quod ipſis conſtructionibus verificatur.

Coroll. VII.

§. 238. Si angulus A ponatur aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus, et trapezio in triangulum BCD abeunte, pro recta EC aequatio eſt hujusmodi:

$$f = \frac{\sqrt{(c^2 p^2 + 2acmhq \pm cp\sqrt{(c^2 p^2 - 4amh(amh - cq))})}}{\sqrt{2}}$$

in qua aequatione eſt $p = - \, \mathit{ſin}. (\beta + d)$ et $q = - \cos. (\beta + d)$, ſed poſito eodem tribus rectis aequali, cadente A ſupra diagonalem BD, et trapezio inverſo prodeunte, habetur haec aequatio:

$$f = \frac{\sqrt{(c^2 q^2 - 2acmhp \mp cq\sqrt{(c^2 q^2 - 4amh(cp + amh))})}}{m\sqrt{2}}$$

exiſtente $p = \mathit{ſin}. (\beta + \lambda)$, $q = - \cos. (\beta + \lambda)$.

Scholion V.

§. 239. Reſolvit haec aequatio penultima novum ad triangula ſpectans problema, nam in

trian-

triangulo totali BCD, non datur nisi latus BD, angulus B, et angulus ECD pars anguli totalis C; in partiali triangulo CBE datur latus BE et angulus adjacens B, in altero partiali CDE datur latus DE et angulus oppositus ACD, et tamen omnia triangula hac methodo tetragonometrica solvuntur.

Coroll. VIII.

§. 240. Si angulus B sit aequalis duobus rectis, lateribus AB, BC in diagonalem AC incidentibus et trapezio in triangulum ACD abeunte, aequatio pro diagonali fit: $f = \frac{cp}{m}$, seu $f\,\mathit{sin.}\,\beta = c\,\mathit{sin.}\,(\beta+\psi)$, unde resultat haec analogia trita trigonometrica: $c : \mathit{sin.}\,\beta = f : \mathit{sin.}(\beta+\psi)$, quo ipso igitur calculus verificatur. Posito vero eodem angulo tribus rectis aequali, cadente vertice B ultra diagonalem AC, intra vel extra dextrum triangulum et trapezio inverso prodeunte, aequatio pro diagonali fit talis:

$$f = \frac{\sqrt{(c^2q^2 - 2acmhp \mp cq\sqrt{(c^2q^2 + 4am(cp - am))}}}{m\sqrt{2}}$$

in qua aequatione est $q = -\,\mathit{cos.}\,(\beta+\psi)$, $p = +\,\mathit{sin.}\,(\beta+\psi)$.

Coroll. IX.

§. 241. Si anguli A et B simul sumti sint duobus rectis aequales, ut prodeat trapezium parallelarum basium AD, BC, fit pro diagonali haec aequatio:

$$f = \frac{\sqrt{(c^2m^2 + 2acmhq \mp cm\sqrt{(c^2m^2 + 4amh(cq - amh))})}}{m\sqrt{2}}$$

$$f = \frac{\sqrt{(c^2m^2 + 2acmhq \mp cm\sqrt{(c^2m^2 + 4amh(cq - amh))})}}{m\sqrt{2}}$$

Si

Si vero ſint tribus rectis aequales habetur haec:

$$f = \frac{\sqrt{(c^2q^2 - 2acm^2h \mp cq\sqrt{(c^2q^2 - 4am^2h(c+ah))})}}{m\sqrt{2}}$$

ubi eſt $q = -cos.\beta, p = m = ſin.\beta$.

Coroll. X.

§. 242. Si angulus B ſit rectus, ſit aequatio talis:

$$f = \frac{\sqrt{(c^2q^2 + 2acmp \pm cq\sqrt{(c^2q^2 - 4am(am - cp))})}}{m\sqrt{2}}$$

Si angulus ACD ſit rectus habetur haec aequatio:

$$f = \frac{\sqrt{(c^2q^2 + 2achp \pm cq\sqrt{(c^2q^2 - 4ah(ah - cp))})}}{\sqrt{2}}$$

denique ſi angulus A ſit rectus habetur:

$$f = \frac{\sqrt{(c^2q^2 + 2acmhp \pm cq\sqrt{(c^2q^2 - 4amh(amh - cp))})}}{m\sqrt{2}}$$

in quarum prima eſt $p = -ſin.(\beta+\psi)$, in ſecunda $= -ſin.(\lambda+\psi)$, in tertia $= -ſin.(\beta+d)$; ſed in prima $q = cos.(\beta+\psi)$, in ſecunda $= cos.(\lambda+\psi)$, in tertia $= cos.(\beta+d)$.

Coroll. XI.

§. 243. Quod ſi ponatur $c^2p^2 == +amh(cq+amh)$ ſignum radicale dextrum aequationis evaneſcit et fit aequatio $f = \frac{\sqrt{(c^2p^2 - 2achmq)}}{m\sqrt{2}}$, vel etiam pro c^2p^2 ſubſtituto ſuo valore, $f == \frac{\sqrt{(ah(cq + 2amh))}}{\sqrt{m}}$, ſive valoribus reſtitutis $f == \frac{\sqrt{(a\,ſin.\lambda(c\,cos.(\beta+\lambda+\psi) + 2a\,ſin.\lambda\,ſin.\beta))}}{\sqrt{ſin.\beta}}$;

 fit

fit autem in hoc casu $c = \frac{2amh(q \pm 1)}{p^2}$.

Si vero ponatur $c^2p^2 = 2acmhq$ fit aequatio:

$$f = \frac{\sqrt{(cp\sqrt{(c^2p^2 - 4amh(cq+amh))})}}{m\sqrt{2}}, \text{ vel}$$

etiam substituto valore ipsius $c^2 p^2$,

$$f = \frac{\sqrt{(cp\sqrt{(-2amh(cq+2amh))})}}{m\sqrt{2}} \text{ quo in casu}$$

$$\text{fit } c = \frac{2amhq}{p^2} = \frac{2a \, fin. \lambda \, fin. \beta \, cos. (\beta + d + \psi)}{fin. (\beta + d + \psi)^2} \text{ vel}$$

$$c \; tang. (\beta + \lambda + \psi) = \frac{2a \, fin. \lambda \, fin. \beta}{fin. (\beta + d + \psi)}, \text{ haec autem}$$

aequatio ne fiat imaginaria debet h vel m, vel q negative sumi, sic enim fiet:

$$f = \frac{\sqrt{(cp\sqrt{(2amh(cq-2amh))})}}{m\sqrt{2}}.$$

Scholion VI.

§. 244. Cum ex problemate decimo tertio sit sinus anguli $D = fin. (\beta + \lambda + \psi) \sqrt{(f^2 - a^2 fin. \lambda^2)} - a \, fin. \lambda \, cos. (\beta + \lambda + \psi)$ et ex scholio hujus problematis constet, hunc sinum ita variari, ut generaliter exprimendus sit hoc modo: $\mp fin. (\pm \beta \pm \psi \pm \lambda) \sqrt{f^2 - a^2 fin. \lambda^2)} \mp a \, fin. \lambda \, cos. (\pm \beta \pm \psi \pm \lambda)$; statim sequitur, aequationem omnium generalissimam hujus XVI. problematis pro omni trapezio tam directo quam inverso esse hujusmodi: $f^2 fin. \beta = \mp c \, fin. \beta \, (\pm \beta \pm \psi \pm \lambda) \sqrt{(f^2 - a^2 fin. \lambda^2)} \mp ac \, fin. \lambda \, cos. (\pm \beta \pm \psi \pm \lambda)$.

Proble-

Problema XVII.

§. 245. In figura quadrilatera propofita Fig. XVII. $ABCD$, inter haec fex: latus $BC=b$, diagonalem $AC=f$, et quatuor angulos $A=\psi$, $B=\lambda$, $ACD=\beta$, et denique $D=\varphi$, aequationem invenire

Solutio.

1) Cum in triangulo dextro habeantur nomina primitiva angulorum D et ACD; a nomine primitivo duorum rectorum π, haec aufero et obtineo nomen derivativum anguli tertii $CAD\ \pi-\varphi-\beta$, quod a nomine primitivo dato ψ, anguli A, iterum aufero et obtineo nomen derivativum anguli CAB, in triangulo finiftro, $\beta+\varphi+\psi-\pi$, cujus itaque finus erit $=-$ *fin.* $(\beta+\varphi+\psi)$.

2) Hinc in triangulo finiftro, ad hanc analogiam pervenio AC: *fin.* $B=BC$: *fin.* CAB, h. e. f: *fin.* $\lambda=b$: $-$ *fin.* $(\beta+\varphi+\psi)$ et confequenter habetur haec aequatio ab omni irrationalitate libera: b *fin.* $\lambda=-f\ (\beta+\varphi+\psi)$. Q. E. I.

Scholion I.

§. 246. Ad hanc aequationem paulo aliter perveniri poterat hoc modo. Quia omnibus angulis in figura impofita funt nomina primitiva excepto angulo ACB, ad obtinendum hujus nomen derivativum, fubduco omnes angulos a quatuor rectis, quorum nomen primitvum 2π, et obtineo nomen quaefitum derivativum $2\pi-\beta-\lambda-\varphi-\psi$, confequenter eft *fin.* ACB $=-$ *fin.* $(\beta+\lambda+\varphi+\psi)$ et *cos.* ACB $=$ *cos.* $(\beta+\lambda+\varphi+\psi)$, ex quibus et ex finu cofi-

cofinuque anguli B formo finum anguli tertii in eodem triangulo, quem igitur invenio effe $= - fin. (\beta + \lambda + \phi + \psi) \; cos. \lambda + cos. (\beta + d + \phi + \psi) \; fin. \lambda = - fin. (\beta + \phi + \psi) = fin. CAB$ ut ante, et ad eandem cum priore analogiam et aequationem pervenio.

Coroll. I.

§. 247. Ex hac aequatione ftatim incurrit in oculos effe latus $BC = b = \frac{-f \, fin. (\beta + \phi + \psi)}{fin. \lambda}$ et diagonalem $AC = f = b \, fin. \lambda : - fin. (\beta + \phi + \psi)$, praeterea angulum $B = \lambda = ang. fin. \frac{-f \, fin. (\beta + \phi + \psi)}{b}$ et cum finus fummae angulorum reliquorum fit $fin. (\beta + \phi + \psi) = \frac{-b \, fin. \lambda}{f}$, erit fumma angulorum ipforum, $\beta + \phi + \psi = ang. \; fin. \; \frac{-b \, fin. \lambda}{b}$ confequenter erit:

$$\text{angulus } \beta = ang. fin. \frac{-b \, fin. \lambda}{b} - \phi - \psi$$

$$\phi = ang. fin. \frac{-b \, fin. \lambda}{b} - \beta - \psi$$

$$\psi = ang. fin. \frac{-b \, fin. \lambda}{b} - \beta - \phi$$

Coroll. II.

§. 248. Singulorum autem angulorum A, D, ACD, finus obtinetur per aequationes quadraticas hoc modo:

fin.

$$\text{fin.}\,\beta = \frac{-\,b\,\text{fin.}\,\lambda\,\text{cos.}\,(\varphi+\psi) \pm \text{fin.}\,(\varphi+\psi)}{\sqrt{(f^2 - b^2\,\text{fin.}\,\lambda^2)}}$$

$$\text{fin.}\,\varphi = \frac{-\,b\,\text{fin.}\,\lambda\,\text{cos.}\,(\beta+\psi) \pm \text{fin.}\,(\beta+\psi)}{\sqrt{(f^2 - b^2\,\text{fin.}\,\lambda^2)}}$$

$$\text{fin.}\,\psi = \frac{-\,b\,\text{fin.}\,\lambda\,\text{cos.}\,(\beta+\varphi) \pm \text{fin.}\,(\beta+\varphi)}{\sqrt{(f^2 - b^2\,\text{fin.}\,\lambda^2)}}.$$

Scholion II.

§. 249. Inter fex cafus, qui in aequatione continentur hujus problematis, nullus datur Tetragonometriae proprius, fed omnes utrique methodo et trigonometricae et tetragonometricae fubfunt; caufa hujus eft, quod omnes anguli revera fint dati, datis illis, qui in figura nomina primitiva habent, cum omnibus a quatuor rectis fubductis, relinquatur angulus qui incognitus effe videbatur ACB, hoc itaque problema revera non nifi fecundario quodam modo ad Tetragonometriam pertinet, cum nihil fuppeditet, quod per Trigonometriam vulgarem non conftet, quoad latus et diagonalem. Solutio vero angulorum expreffionis brevitate et elegantia parum vel nihil Trigonometricis cedunt.

Coroll. III.

§. 250. Quod fi anguli D, et ACD fint fimul fumti uni recto aequales prodit haec aequatio: $b\,\text{fin.}\,d = -\,f\,\text{cos.}\,\psi$ et hinc haec analogia: $f : \text{fin.}\,\lambda = b : -\,\text{cos.}\,\psi$. Si anguli A et ACD fint fimul uni recto aequales, prodit haec aequatio: $b\,\text{fin.}\,\lambda = -\,f\,\text{cos.}\,\varphi$, et hinc haec analogia: $f : \text{fin.}\,\lambda = b : -\,\text{cos.}\,\varphi$; Si denique anguli A et D fint fimul fumti uni recto aequales; prodit

b fin.

$b\,fin.\,\lambda = - f\,cos.\,\beta$, et hinc haec analogia: $f : fin.\,d = b : - cos.\,\beta$, quae analogiae totidem fuppeditant theoremata.

Coroll. IV.

§. 251. Si anguli A, et B, ponantur fimul duobus rectis aequales; aequatio primo inventa fit: $b\,fin.\,\psi = - f\,fin.\,(\beta + \phi + \psi)$, unde elicitur haec aequatio: $tang.\,\psi = \frac{- f\,fin.\,(\beta + \phi)}{b + f\,cos.\,(\beta + \phi)}$ vel etiam $tang.\,\lambda = \frac{- f\,tang.\,(\beta + \phi)}{f + b\,fec.\,(\beta + \phi)}$.

Coroll. V.

§. 252. Si anguli diagonaliter oppofiti B et D, fint fimul fumti duobus rectis aequales, prodeunte fic trapezio circulo infcriptibile, prodit haec aequatio: $b\,fin.\,\phi = - f\,fin.\,(\beta + \phi + \psi)$, ex qua obtinetur haec: $tang.\,\phi = \frac{- f\,fin.\,(\beta + \psi)}{b + f\,cos.\,(\beta + \psi)} = \frac{- f\,tang.\,(\beta + \psi)}{f + b\,fec.\,(\beta + \psi)}$.

Coroll. VI.

§. 253. Si angulus A fit duobus rectis aequalis lateribus AB, AD in diagonalem BD incidentibus et trapezio in triangulum BCD abeunte, prodit haec aequatio: $b\,fin.\,\lambda = f\,fin.\,(\beta + \psi)$, unde haec analogia: $fin.\,(\beta + \phi) : b = fin.\,\lambda : f$.

Schol.

Scholion III.

§. 254. Hic nova quadam ratione ſolvitur problema ad Trigonometriam ſpectans, nam in triangulo totali DBC, dantur tres anguli B, D, et partialis ECD, et latus BC; in partiali EBC, non dantur niſi duo, latus BC, et angulus adjacens B, in partiali ECD non dantur niſi duo anguli, D et ACD, recta vero EC hac analogia nova quadam ratione detegitur, ex uno latere et tribus angulis.

Coroll. VII.

§. 255. Sed anguli etiam D, et ECD nova quadam ratione obtinentur, fit enim in hoc caſu ex corollario II. $\text{ſin.}\ \beta = b\ \text{ſin.}\ \lambda\ \cos.\ \phi \mp \text{ſin.}\ \phi \sqrt{(f^2 - b^2\ \text{ſin.}\ \lambda^2)}$ per aequationem primam, et per ſecundam habetur $\text{ſin.}\ \phi = b\ \text{ſin.}\ \lambda\ \cos.\ \beta \mp \text{ſin.}\ \beta \sqrt{(f^2 - b^2\ \text{ſin.}\ \lambda^2)}$.

Coroll. VIII.

§. 256. Si angulus B ſit rectus fit $b = -f\ \text{ſin.}\ (\beta + \phi + \psi)$, ſi A rectus, fit $b\ \text{ſin.}\ \lambda = -f\ \cos.\ (\beta + \phi)$, ſi D rectus, fit $b\ \text{ſin.}\ \lambda = -f\ \cos.\ (\beta + \psi)$; ſi denique ACD rectus, fit: $b\ \text{ſin.}\ d = -f\ \cos.\ (\phi + \psi)$; Si vero angulus B, ſit tribus rectis aequalis, cadente vertice B, ultra diagonlem intra vel extra dextrum triangulum, et trapezio inverſo prodeunte, fit $b = f\ \text{ſin.}\ (\beta + \phi + \psi)$. Si A ſit aequalis tribus rectis, cadente vertice A ſupra diagonalem BD, intra vel extra triangu- BCD, et trapezio inverſo denuo prodeunte, fit $b\ \text{ſin.}\ \lambda = f\ \cos.\ (\beta + \phi)$; Si D ſit tribus rectis aequalis, cadente vertice D, intra vel extra ſini-

ſiniſtrum triangulum, et trapezio inverſo prodeunte habetur: $b\ \mathit{fin.}\ \lambda = f\ \mathit{cos.}\ (\beta+\psi)$. Quod ſi denique ponatur angulus ACD aequalis tribus rectis, cadente latere CD, ſupra latus CB, et angulum rectum cum diagonali AC faciente, et trapezio totaliter inverſo prodeunte, ſit $b\ \mathit{fin.}\ \lambda = f\ \mathit{cos.}\ (\varphi+\psi)$; idem prodiret ponendo angulum ACD negativum et uni recto aequalem.

Coroll. IX.

§. 257. Ponendo angulum A rectum vel tribus rectis aequalem; aequationes corollarii II. ita mutantur ut fiat: $\mathit{fin.}\ \beta = \mp\ b\ \mathit{fin.}\ \lambda\ \mathit{fin.}\ \varphi \pm \mathit{cos.}\ \varphi\ \surd(f^2 - b^2 \mathit{fin.}\ \lambda^2)$; $\mathit{fin.}\ \varphi = \mp\ b\ \mathit{fin.}\ \lambda\ \mathit{fin.}\ \beta \pm \mathit{cos.}\ \beta\ \surd(f^2 - b^2 \mathit{fin.}\ \lambda^2)$; angulum D rectum vel tribus rectis aequalem ponendo ſit $\mathit{fin.}\ \beta = \mp\ b\ \mathit{fin.}\ \lambda\ \mathit{fin.}\ \psi \pm \mathit{cos.}\ \psi\ \surd(f^2 - b^2 \mathit{fin.}\ \lambda^2)$, et $\mathit{fin.}\ \psi = \mp\ b\ \mathit{fin.}\ \lambda\ \mathit{fin.}\ \varphi \pm \mathit{cos.}\ \varphi\ \surd(f^2 - b^2\ \mathit{fin.}\ \lambda^2)$, ſed ponendo angulum B rectum vel tribus rectis aequalem habentur hae aequationes:

$$\mathit{fin.}\ \beta = \mp\ b\ \mathit{cos.}\ (\varphi+\psi) \pm \mathit{fin.}\ (\varphi+\psi)\ \surd(f^2 - b^2)$$
$$\mathit{fin.}\ \varphi = \mp\ \mathit{cos.}\ (\beta+\psi) \pm \mathit{fin.}\ (\beta+\psi)\ \surd(f^2 - b^2)$$
$$\mathit{fin.}\ \psi = \mp\ b\ \mathit{cos.}\ (\beta+\psi) \pm \mathit{fin.}\ (\beta+\varphi)\ \surd(f^2 - b^2)$$

Angulum ACD rectum vel tribus rectis aequalem ponendo ſit: $\mathit{fin.}\ \varphi = \mp\ b\ \mathit{fin.}\ \lambda\ \mathit{fin.}\ \psi \pm \mathit{cos.}\ \psi\ \surd(f^2 - b^2 \mathit{fin.}\ \lambda^2)$, $\mathit{fin.}\ \psi = \mp\ b\ \mathit{fin.}\ \lambda\ \mathit{fin.}\ \varphi \pm \mathit{cos.}\ \varphi\ \surd(f^2 - b^2 \mathit{fin.}\ \lambda^2)$. Poſitione anguli recti valent ſigna ſuperiora, ſed poſitione anguli tribus rectis aequalis, inferiora. His denique poſitionibus hanc addo; Si omnes anguli A, B, D, ACD ſint ſimul ſumti tribus rectis aequales, ut fiat angulus ACB recto aequalis, fiet $f = b\ \mathit{tang.}\ \lambda$, uti per vulgarem Trigonometriam conſtat.

Scholion IV.

§. 258. Si trapezium sit partialiter inversum, ut totum triangulum ADC cadat intra triangulum ABC, habetur haec aequatio: $b\ fin.\ \lambda = \mp f\ fin.\ (\psi - \beta - \varphi)$, eadem reproducitur, si trapezium sit totaliter inversum, ut latus AD sit intra latus AB, et latus CD extra latus CB. Quod si latus AD sit extra latus AB, latus autem CD intra latus CB, habetur haec aequatio: $b\ fin.\ \lambda = f\ fin.\ (\beta + \varphi + \psi)$. Cum igitur plures variationes occurrere nequeant, erit aequatio generalis pro omni trapezio tam directo quam inverso talis: $b\ fin.\ \lambda = \mp f\ fin.\ (\psi \pm \beta \pm \varphi)$.

Problema XVIII.

§. 259. In figura quadrilatera proposita $ABCD$ inter haec sex: latus $AB=a$, $BC=b$, diagonalem $AC=f$, atque angulos: $A=\psi$, $ACD=\beta$ et denique $D=\varphi$, aequationem invenire. Fig. XVIII.

Solutio.

Cum in resolutione problematis praecedentis inventum sit nomen derivativum anguli CAB esse $\beta + \varphi + \psi) - \pi$; sequitur esse $cos.\ CAB = - cos.\ (\beta + \varphi + \psi)$ et per (Euclid. II. 11.) formo hanc aequationem in triangulo sinistro: $BC^2 = AB^2 + AC^2 - 2Ac\ Ab$; ab angulo B in diagonalem AC perpendiculo Bb demisso; fiet igitur symbolis substitutis aequatio talis: $b^2 = a^2 + f^2 + 2af\ cos.\ (\beta + \varphi + \psi)$ ab omni irrationalitate libera.

Coroll.

Coroll. I.

§. 260. Ex hac aequatione ſtatim incurrit in oculos, eſſe latus $BC = b = \surd(a^2 + f^2 + 2af\ cos.\ (\beta + \phi + \psi))$. Sed eſt etiam latus $AB = a = -f\ cos.\ (\beta + \phi + \psi) \pm \surd\ (b^2 - f^2\ fin.\ (\beta + \phi + \psi^2))$ et diagonalis $AC = f = -a\ cos.\ (\beta + \phi + \psi) \pm \surd\ (b^2 - a^2\ fin.\ (\beta + \phi + \psi)^2)$.

Coroll. II.

§. 261. Ex eadem aequatione ſinguli auguli facillime et quidem ſimiliter exprimuntur, cum enim ſit $cos.\ (\beta + \phi + \psi) = \frac{b^2 - a^2 f^2}{2af}$; habetur ſumma omnium angulorum hoc modo: $\beta + \phi + \psi = ang.\ cos.\ \frac{(b^2 - a^2 - f^2)}{2af}$ atque hinc erit

$$\text{angulus } \beta = ang.\ cos. \frac{(b - a^2 - f^2)}{2af} - \phi - \psi$$

$$\text{angulus } \phi = ang.\ cos. \frac{(b^2 - a^2 - f^2)}{2af} - \beta - \psi$$

$$\text{et denique angulus } \psi = ang.\ cos. \frac{(b^2 - a^2 - f^2)}{2af} - \beta - \phi.$$

Coroll. III.

§. 262. Singulorum autem angulorum ſinus per aequationes quadraticas affectas prorſus inter ſe ſimiles exprimuntur, quia ſinguli anguli aequationi eodem modo iuſunt. Eſt itaque

ſin.

$$\text{fin. } \beta = \frac{(a^2+f^2-b^2)\,\text{fin.}(\varphi+\psi) \pm \cos.(\varphi+\psi)}{2af}$$
$$\frac{\Gamma(4a^2-f^2-(b^2-a^2-f^2)^2)}{2af}$$
$$\text{fin. } \varphi = \frac{(a^2+f^2-b^2)\,\text{fin.}(\beta+\psi) \pm \cos.(\beta+\psi)}{2af}$$
$$\frac{\Gamma(4a^2f^2-(b^2-a^2-f^2)^2)}{2af}$$
$$\text{fin. } \psi = \frac{(a^2+f^2-b^2)\,\text{fin.}(\beta+\varphi) \pm \cos.(\beta+\varphi)}{2af}$$
$$\frac{\Gamma(4a^2f^2-(b^2-a^2-f^2)^2)}{2af}$$

Scholion I.

§. 263. Uti antecedens problema ita etiam hoc nullum continet cafum Tetragonometriae proprium, fed omnes fex cafus Tetragonometriae pariter ac Trigonometriae fubfunt, ita tamen ut folutiones tetragonometricae, quas dedi, quia ex aequatione breviter et facile confequntur, folutionibus trigonometricis parum vel nihil cedunt.

Coroll. IV.

§. 264. Ponantur finguli anguli aequationem ingredientes recti; et exiftente angulo A recto, fit aequatio pro diagonali $f = a\,\text{fin.}\,(\beta+\varphi) \pm \Gamma(b^2-a^2\cos.\,(\beta+\varphi)^2)$; exiftente angulo D recto, $f = a\,\text{fin.}\,(\beta+\psi) \pm \Gamma(b^2-a^2\cos.\,(\beta+\psi)^2)$; exiftente ACD recto, $f = a\,\text{fin.}\,(\varphi+\psi) \pm \Gamma(b^2-a^2\cos.\,(\varphi+\psi)^2)$; fed pofito angulo A tribus rectis aequali, cadente

vertice A ſupra diagonalem BD, intra vel extra triangulum BCD, et prodeunte trapezio inverſo, fit aequatio pro diagonali $f = -a\,\text{ſin.}(\beta+\varphi) \pm \sqrt{(b^2 - a^2 \cos.(\beta+\varphi)^2)}$; poſito D tribus rectis aequali, cadente D ad ſiniſtram diagonalis intra vel extra triangulum ABC, et prodeunte rurſus trapezio inverſo, fit $f = -a\,\text{ſin.}\,(\beta+\psi) \pm \sqrt{\,b^2 - a^2 \cos.\,(\beta+\psi)^2)}$, denique poſito angulo ACD tribus rectis aequali, cadente latere CD ſupra vel infra latus BC, et prodeunte ſic trapezio inverſo, habetur pro diagonali: $AC = f = -a\,\text{ſin.}(\varphi+\psi) \pm \sqrt{(b^2 - a^2 \cos.(\varphi+\psi)^2)}$; idem prodit ponendo angulum ACD negativum et uni recto aequalem. Hae poſitiones non niſi ſignis diſcrepant a prioribus ut ſupra.

Coroll. V.

§. 265. Singulis angulis ut prius uni recto vel tribus rectis aequalibus ponendis, aequationes pro ſinibus ex Coroll. III. hoc modo mutantur, et quidem ponendo angulum A

$$\text{ſin.}\,\beta = \frac{\pm(a^2+f^2-b^2)\cos.\varphi \mp \text{ſin.}\varphi \sqrt{(4a^2f^2-(b^2-a^2-f^2)^2)}}{2\,a\,f}$$

$$\text{ſin.}\,\varphi = \frac{\pm(a^2+f^2-b^2)\cos.\beta \mp \text{ſin.}\beta \sqrt{(4a^2f^2-(b^2-a^2-f^2)^2)}}{2\,a\,f}$$

ſed ponendo angulum D habentur hae aequationes:

$$\text{ſin.}\,\beta = \frac{\pm(a^2+f^2-b^2)\cos.\psi \mp \text{ſin}\,\psi \sqrt{(4a^2f^2-(b^2-a^2f^2)^2)}}{2\,a\,f}$$

$$\text{ſin.}\,\psi = \frac{\pm(a^2+f^2-b^2)\cos.\beta \mp \text{ſin.}\beta \sqrt{(4a^2f^2-(b^2-a^2-f^2)^2)}}{2\,a\,f}$$

denique ponendo angulum ACD prodeunt

ſin.

$$\sin.\varphi = \frac{\pm(a^2+f^2-b^2)\cos.\psi \mp \sin\psi \surd(4a^2f^2-(b^2-a^2-f^2)^2)}{2af}$$

$$\sin.\psi = \frac{\pm(a^2+f^2-b^2)\cos.\varphi \mp \sin.\varphi \surd(4a^2f^2-(b^2-a^2-f^2)^2)}{2af}$$

signis superioribus pro positione anguli recti, inferioribus pro positione anguli tribus rectis aequali valentibus.

Coroll. VI.

§. 266. Si anguli A et D simul sumti ponantur aequales duobus rectis, prodeunte trapezio parallelarum basium AB, DC, aequatio pro diagonali fit $f = a \cos.\beta \pm \surd(b^2 - a^2 \sin.\beta^2)$; pro latere AB fit $a = f\cos.\beta \pm \surd(b^2 - a^2 \sin.\beta^2)$; pro latere BC fit $b = \surd(a^2 - f^2 - 2af\cos.\beta)$; sed aequatio pro $\sin.\beta = \frac{\surd(4a^2f^2 - (b^2 - a^2 - f^2))}{2af}$.

Coroll. VII.

§. 267. Si angulus A sit duobus rectis aequalis, lateribus AB, AD in diagonalem BD incidentibus et trapezio in triangulum BCD abeunte, fit aequatio pro diagonali $f = a\cos.(\beta+\varphi) \pm \surd(b^2 - f^2 \sin.(\beta+\varphi)^2)$; sed aequationes ex Coroll. III. pro sinibus fiunt:

$$\sin.\beta = \frac{-(a^2+f^2-b^2)\sin.\varphi \mp \cos\varphi \surd(4a^2f^2-(b^2-a^2-f^2)^2)}{2af}$$

$$\sin.\varphi = \frac{-(a^2+f^2-b^2)\sin.\beta \mp \cos.\beta \surd(4a^2f^2-(b^2-a^2-f^2)^2)}{2af}$$

denique ipsi anguli evadunt hujusmodi:

$$\beta = \text{ang. cos.} \frac{(b^2-a^2-f^2)}{2af} - \varphi; \text{ nec non}$$

 $\varphi =$

$\varphi = ang.\ cos.\ \frac{(b^2 - a^2 - f^2)}{2af} - \beta$, denique pro latere $BC = b = \surd(a^2 + f^2 - 2af\,cos.(\beta + \varphi))$ et pro latere $AB = a = f\,cos.(\beta + \varphi) \pm \surd(b^2 - f^2 fin.(\beta + \varphi)^2)$.

Scholion II.

§. 268. Harum aequationum prima, nova ratione ſolvit problema ad triangula ſpectans, nam in triangulo totali BCD datur latus unum BC, et angulus oppoſitus D, pars EB lateris BD, et pars ACD anguli C, et tamen ex his invenitur recta EC. In triangulo partiali EBC non dantur niſi duo latera EB, BC, nec in triangulo ECD, niſi duo anguli.

Coroll. VIII.

§. 269. Si anguli A et ACD ſimul ſumti duobus rectis aequales ponantur, fit pro diagonali haec aequatio: $f = a\,cos.\,\varphi \pm \surd(b^2 - a^2 fin.\,\varphi^2)$, pro latere AB fit $a = f\,cos.\,\varphi \pm \surd(b^2 - f^2 fin.\,\varphi^2)$, pro latere BC fit $b = \surd(a^2 + f^2 - 2af\,cos.\,\varphi^2)$; ſed in hoc caſu fit $fin.\,\varphi = \frac{\surd(4a^2f^2 - (b^2 - a^2 - f^2)^2)}{2af}$

Si vero uni recto vel tribus rectis aequales ponantur fit ex Coroll. I. $f = \pm a\,fin.\,\varphi \pm \surd(b^2 - a^2 cos.\,\varphi^2)$; $a = \pm f\,fin.\,\varphi \pm \surd(b^2 - f^2\ cos.\,\varphi^2)$; $b = \surd(a^2 + f^2 \pm 2af\,fin.\,\varphi)$. Sed in hoc caſu fit $fin.\,\varphi = \pm (a^2 + f^2 - b^2) : 2af$. Fit igitur in caſu priore ipſe angulus $\varphi = ang.\ fin.\ (a^2 + f^2 - b^2) : 2af$, in caſu poſteriore $\varphi = ang.\,fin.\,(b^2 - a^2 - f^2) : 2af$, quod cum aequatione in Coroll. II. inventa con-

consentit. $\varphi = ang.\, cos. \frac{(b^2-a^2-f^2)}{2af} - \beta - \psi$ est enim positum $\beta + \psi = \frac{\pi}{2}$ vel $\beta + \psi = \frac{3\pi}{2}$, quare facta transpositione fit $\frac{\pi}{2} + \psi = ang.\, cos. \frac{(b^2-a^2-f^2)}{2af}$ vel $\frac{3\pi}{2} + \varphi = ang.\, cos. \frac{(b^2-a^2-f^2)}{2af}$ et consequenter fit in casu priore $cos. \frac{(3\pi+\varphi)}{2} = \frac{b^2-a^2-f^2}{2af}$ et hinc $fin.\ \varphi = \frac{b^2-a^2-f^2}{2af}$ et consequenter $\varphi = ang.\, fin. \frac{(a^2+f^2-b^2)}{2af}$ et in posteriore uti prius.

Coroll. IX.

§. 270. Si anguli A et D, ponantur simul sumti tribus rectis aequales; prodit pro diagonali haec aequatio: $f = -a\, fin. \beta \pm \surd(b^2 - a^2 cos. \beta^2)$; sed pro latere AB fit $a = -f\, fin. \beta \pm \surd(b^2 - f^2 cos. \beta^2)$; denique fit pro latere $BC = b = \surd(a^2 + f^2 + 2af\, fin. \beta)$; sed in hoc casu fit, $fin. \beta = (b^2 - a^2 - f^2) : 2af$. Si anguli D et ACD sint simul sumti tribus rectis aequales, quod fieri potest cadente vertice D ad sinistram diagonalis intra vel extra triangulum ABC, et trapezio inverso prodeunte, fit, $f = -a\, fin. \psi \pm \surd(b^2 - a^2\ cos. \psi^2)$; $a = -f\, fin. \psi \pm \surd(b^2 - f^2\ cos. \psi^2)$; $b = \surd(a^2 + f^2 + 2af\, fin. \psi^2)$, et denique $fin. \psi = \frac{a^2+f^2-b^2}{2af}$.

 Schol.

Scholion III.

§. 271. Si trapezium ita ſit partialiter inverſum ut totum triangulum ADC cadat intra triangulum ABC habetur haec aequatio: $b^2 = a^2 + f^2 \pm 2af\,cos.(\psi-\beta-\phi)$. Sin autem latus AD cadat extra latus AB, CD vero intra CB, habetur haec: $b^2 = a^2 + f^2 \pm 2af\,cos.(\beta+\phi+\psi)$; nec praeter has variationes plures occurrere poſſunt, quare generalis aequatio hanc habet formam: $b^2 = a^2 + f^2 \pm 2af(\psi \pm \beta \pm \phi)$.

Problema XIX.

Fig. XIX. §. 272. In figura quadrilatera propoſita $ABCD$, inter haec ſex: latus $AB = a$, diagonalem $AC = f$, et quatuor angulos, $A = \psi$, $ACD = \beta$, $B = \lambda$, et denique $D = \phi$; aequationem invenire.

Solutio.

Cum e ſcholio primo problematis XVII. ſit inventus $fin.\,ACB = -(\beta+\lambda+\phi+\psi)$; in triangulo ſiniſtro ſtatim pervenio ad hanc analogiam: $AC : fin.\,B = AB : fin.\,ACB$, h. e. $f : fin.\,\lambda = a : -fin.(\beta+\lambda+\phi+\psi)$, obtinetur haec aequatio ab omni irrationalitate libera: $a\,fin.\,\lambda = f\,fin.(\beta+\lambda+\phi+\psi)$. Q. E. I.

Coroll. I.

§. 273. Ex hac aequatione primo ſtatim intuitu eſt latus $AB = a = \frac{-f\,fin.(\beta+\lambda+\phi+\psi)}{fin.\,\lambda}$, nec non diagonalis $AC = f = a\,fin.\lambda : -fin.(\beta+\lambda+\phi+\psi)$, ſed pro angulo B inveniendo habetur haec aequa-

aequatio: $tang.\ \lambda = \frac{-f\ \text{fin.}\ (\beta+\varphi+\psi)}{a+f\ \cos.\ (\beta+\varphi+\psi)} =$
$= \frac{-f\ tang.\ (\beta+\varphi+\psi)}{f+a\ \text{fec.}\ (\beta+\varphi+\psi)}$.

Coroll. II.

§. 274. Cum ex hac aequatione fit fumma omnium angulorum A, ACD, et D, $\beta+\varphi+\psi = ang.\ \text{fin.} \frac{-a\ \text{fin.}\ \lambda}{f} - \lambda$; habetur

angulus $\psi = ang.\ \text{fin.} \frac{-a\ \text{fin.}\ \lambda}{f} - \lambda - \beta - \varphi$

angulus $\varphi = ang.\ \text{fin.} \frac{-a\ \text{fin.}\ \lambda}{f} - \lambda - \beta - \psi$

angulus $\beta = ang.\ \text{fin.} \frac{-a\ \text{fin.}\ \lambda}{f} - \lambda - \varphi - \psi$.

Coroll. III.

§. 275. Sed trium horum angulorum finus per aequationes fimiles exprimuntur hoc modo:

$\text{fin.}\ \beta = \frac{-a\ \text{fin.}\ \lambda\ \cos.\ (\lambda+\varphi+\psi)}{f} \pm$
$\frac{\text{fin.}\ (\lambda+\varphi+\psi)\ \surd(f^2 - a^2\ \text{fin.}\ \lambda^2)}{f}$

$\text{fin.}\ \varphi = \frac{-a\ \text{fin.}\ \lambda\ \cos.\ (\lambda+\beta+\psi)}{f} \pm$
$\frac{\text{fin.}\ (\lambda+\beta+\psi)\ \surd(f^2 - a^2\ \text{fin.}\ \lambda^2)}{f}$

$$\mathit{fin.}\,\psi = \frac{-a\,\mathit{fin.}\,\lambda\;\mathit{cos.}\,(\lambda+\beta+\varphi)}{f} \pm \frac{\mathit{fin.}\,(\lambda+\beta+\varphi)\,\surd\,(f^2 - a^2\,\mathit{fin.}\,\lambda^2)}{f}.$$

Scholion I.

§. 276. Quemadmodum duo proxime praecedentia problemata; ita hoc etiam problema nullum continet casum Tetragonometriae proprium, sed omnes sex casus utrique methodo et Trigonometricae et Tetragonometricae subsunt; quoad latus et diagonalem solutio tetragonometrica trigonometricae fere praestat, cum omnes sex determinationes simplicem analogiam trigonometricam ingrediantur. Similiter solutiones, quibus anguli inveniuntur, trigonometricis parum vel nihil cedunt.

Coroll. IV.

§. 277. Si angulus A sit rectus fit $f = a\,\mathit{fin.}\,\lambda : -\mathit{cos.}\,(\beta+\lambda+\varphi)$, si B rectus sit, $f = a : -\mathit{cos.}\,(\beta+\varphi+\psi)$, si D rectus sit, $f = a\,\mathit{fin.}\,\lambda : -\mathit{cos.}\,(\beta+\lambda+\psi)$, denique si ACD rectus; fit, $f = a\,\mathit{fin.}\,d : -\mathit{cos.}\,(\lambda+\varphi+\psi)$; sed in casu primo fit $a = -f\,\mathit{cos.}\,(\beta+\lambda+\varphi) : \mathit{fin.}\,\lambda$, in casu secundo fit $a = -f\,\mathit{cos.}\,(\beta+\varphi+\psi)$, in casu tertio fit $a = -f\,\mathit{cos.}\,(\beta+\lambda+\psi) : \mathit{fin.}\,\lambda$, in quarto casu fit, $a = -f\,\mathit{cos.}\,(\lambda+\varphi+\psi) : \mathit{fin.}\,d$.

Coroll. V.

§. 278. Si angulus A sit tribus rectis aequalis cadente vertice D supra diagonalem BD, intra

intra vel extra triangulum BCD, et trapezio inverſo prodeunte; fit $f = a\, fin.\lambda : cos.(\beta+\lambda+\varphi)$, ſed $a = f\, cos.(\beta+\lambda+\varphi) : fin.\lambda$. Si angulus B vel D ſit tribus rectis aequalis, cadente in caſu priore B ad dextram, in poſteriore D ad ſiniſtram diagonalis AC, et in caſu utroque trapezio inverſo prodeunte; fit in caſu priore $f = -a : cos.(\beta+\varphi+\psi)$, ſed $a = -f\, cos.(\beta+\varphi+\psi)$; in poſteriore, $f = a\, fin.\, d : cos.(\beta+\lambda+\psi)$; $a = \frac{-f\, cos.(\beta+\lambda+\psi)}{fin.\lambda}$. Denique ſi angulus ACD tribus rectis aequalis ponatur, cadente latere CD ſupra vel infra latus BC, et trapezio totaliter inverſo prodeunte fit, $f = a\, fin.\lambda : cos.(\lambda+\varphi+\psi)$, ſed $a = f\, cos.(\lambda+\varphi+\psi) : fin.\lambda$. Idem obtinetur ponendo angulum ACD negativum et ſimul rectum. Sed hae aequationes non niſi ſignis differunt ab illis coroll. praecedentis.

Coroll. VI.

§. 279. Sed ponendo angulo A recto vel tribus rectis aequali fit:

$$fin.\beta = \frac{\pm a\, fin.\lambda\, fin.(\lambda+\varphi) \mp cos.(\lambda+\varphi)\sqrt{(f^2 - a^2 fin.\lambda^2)}}{f},$$

$$fin.\varphi = \frac{\pm a\, fin.\lambda\, fin.(\beta+\lambda) \mp cos.(\beta+\lambda)\sqrt{(f^2 - a^2 fin.\lambda^2)}}{f};$$

ſed ponendo angulo D recto vel tribus rectis aequali habetur:

$$fin.\beta = \frac{\pm a\, fin.\lambda\, fin.(\lambda+\psi) \mp cos.(\lambda+\psi)\sqrt{(f^2 - a^2 fin.\lambda^2)}}{f}$$

$$fin.\psi = \frac{\pm a\, fin.\lambda\, fin.(\lambda+\beta) \pm cos.(\lambda+\beta)\sqrt{(f^2 - a^2 fin.\lambda^2)}}{f}.$$

 Ponen-

Ponendo angulo B recto vel tribus rectis aequali habentur hae aequationes:

$$fin.\,\beta = \frac{a\,fin.\,(\varphi+\psi) \mp cos.\,(\varphi+\psi)\,\sqrt{(f^2-a^2)}}{f},$$

$$fin.\,\varphi = \frac{a\,fin.\,(\beta+\psi) \mp cos.\,(\beta+\psi)\,\sqrt{(f^2-a^2)}}{f} \text{ et}$$

$$fin.\,\psi = \frac{a\,fin.\,(\beta+\varphi) \mp cos.\,(\beta+\varphi)\,\sqrt{(f^2-a^2)}}{f}.$$

Denique ponendo angulo ACD recto vel tribus rectis aequali habentur hae aequationes:

$$fin.\varphi = \frac{\pm a\,fin.\lambda\,fin.(\lambda+\psi) \mp cos.(\lambda+\psi)\sqrt{(f^2-a^2 fin.\lambda^2)}}{f}$$

$$fin.\psi = \frac{\pm a\,fin.\lambda\,fin.(\lambda+\varphi) \mp cos.(\lambda+\varphi)\sqrt{(f^2-a^2 fin.\lambda^2)}}{f}$$

Sed in casu primo fit, $tang.\,\lambda = \frac{f\,cos.\,(\beta+\varphi)}{f\,fin.\,(\beta+\varphi)-a}$, vel etiam $tang.\,\lambda = \frac{f\,cot.\,(\beta+\varphi)}{f-a\,cofec.\,(\beta+\varphi)}$, sed in casu secundo fit $tang.\,\lambda = \frac{f\,cos.\,(\beta+\psi)}{f\,fin.\,(\beta+\psi)-a} = \frac{f\,cot.\,(\beta+\psi)}{f-a\,cofec.\,(\beta+\psi)}$ denique fit in quarto casu $tang.\,\lambda = \frac{f\,cos.\,(\varphi+\psi)}{f\,fin.\,(\varphi+\psi)-a} = \frac{f\,cot.\,(\varphi+\psi)}{f-a\,cofec.\,(\varphi+\psi)}$.

Coroll. VII.

§. 280. Si angulus A fit duobus rectis aequalis, lateribus AB, AD, in diagonalem BD incidentibus, et trapezio in triangulum abeunte, fit $f = a\,fin.\lambda : fin.\,(\beta+\lambda+\varphi)$, et $a = f\,fin.(\beta+\lambda+\varphi) : fin.\lambda$. Sed

Sed in hoc casu habentur pro angulis hae aequationes $ang.\ \mathit{fin}.\frac{a\ \mathit{fin}.\ \lambda}{f} - \lambda - \beta = \phi$, et $\beta = ang.\ \mathit{fin}.\frac{a\ \mathit{fin}.\ \lambda}{f} - \lambda - \phi$, et pro finibus,

$$\mathit{fin}.\beta = \frac{a\ \mathit{fin}.\lambda\ cos.(\lambda+\phi) \mp \mathit{fin}.(\lambda+\phi)\surd(f^2 - a^2\mathit{fin}.\lambda^2)}{f}$$

$$\mathit{fin}.\phi = a\ \mathit{fin}.\lambda\ cos.(\lambda+\beta) \mp \mathit{fin}.(\lambda+\beta)\surd(f^2 - a^2\mathit{fin}.\lambda^2)$$

denique fit $tang.\ \lambda = \frac{f\ \mathit{fin}.\ (\beta+\phi)}{a - f\ cos.\ (\beta+\phi)} = \frac{f\ tang.\ (\beta+\phi)}{a\ \mathit{fec}.\ (\beta+\phi) - f}$.

Scholion II.

§. 281. Solvitur hic nova quadam ratione ad triangula spectans problema, quando quaeritur recta EC; nam in triangulo totali datur EB pars lateris BD et anguli adjacentis B, D, et anguli C, pars ECD; in triangulo partiali EBC, non dantur nisi duo, latus EB, et angulus adjacens B, in altero partiali ECD non dantur nisi anguli D, et ECD, et tamen omnia triangula solvuntur. Caeterae determinationes quatuor etiam nova quadam ratione inveniuntur.

Coroll. VIII.

§. 282. Si anguli A et D, ponantur simul sumti duobus rectis aequales, ut prodeat trapezium parallelarum basium AB, CD, fit $f = a\ \mathit{fin}.\ \lambda : \mathit{fin}.\ (\beta + \lambda)$ et

tang.

$tang.\ d == \frac{f\,fin.\,\beta}{a - f\,cos.\,\beta} = \frac{f\,tang.\,\beta}{a\,fec.\,\beta - f}$ nec non

$fin.\beta == \frac{a\,fin.\,\lambda\,cos.\,\lambda \mp fin.\,\lambda\,\surd(f^2 - a^2\,fin.\,\lambda^2)}{f}$; Si autem anguli A et B fimul fumti aequales fint duobus rectis, prodeunte trapezio parallelarum bafium AD, BC, fit, $f == a\,fin.\,\lambda : fin.\,(\beta + \varphi)$; fed in hoc cafu fit

$fin.\,\beta = \frac{a\,fin.\lambda\,cos.\varphi \mp fin.\varphi\,\surd(f^2 - a^2 fin.\lambda^2)}{f}$ vel etiam

$fin.\beta = \frac{a\,fin.\psi\,cos.\varphi \mp fin.\psi\,\surd(f^2 - a^2 fin.\psi^2)}{f}$, nec non

$fin.\varphi = \frac{a\,fin.\lambda\,cos.\beta \mp fin.\beta\,\surd(f^2 - a^2 fin.\lambda^2)}{f}$, vel etiam

$fin.\varphi = \frac{a\,fin.\psi\,cos.\,\beta \mp fin.\,\beta\,\surd(f^2 - a^2 fin.\psi^2)}{f}$. Quod

fi anguli diagonaliter oppofiti B, D, fimul fumti, duobus rectis aequales ponantur, prodeunte trapezio circulo infcriptibile, fit; $f = a\,fin.\,\lambda : fin.\,(\beta + \psi)$ vel $f = a\,fin.\,\varphi : fin.\,(\beta + \psi)$. Fit autem

$fin.\,\beta = \frac{a\,fin.\lambda\,cos.\psi \mp fin.\psi\,\surd(f^2 - a^2 fin.\lambda^2)}{f}$ vel etiam

$fin.\,\beta = \frac{a\,fin.\,\varphi\,cos.\,\psi \mp fin.\,\psi\,\surd(f^2 - a^2\,fin.\,\varphi^2)}{f}$ atque

$fin.\psi = \frac{a\,fin.\,\lambda\,cos.\,\beta \mp fin.\,\beta\,\surd(f^2 - a^2\,fin.\,\lambda^2)}{f}$ vel

$fin.\,\psi = \frac{a\,fin\,\varphi\,cos.\,\beta \mp fin.\,\beta\,\surd(f^2 - a^2\,fin.\,\varphi^2)}{f}$. Sed

ex Corollario II. fit in cafu primo angulus ACD

$ACD = \beta = ang.\,fin.\frac{+a\,fin.\,\lambda}{f} - \lambda$. Sed in cafu fecundo fit, $\beta = ang.\,fin.\frac{+a\,fin.\,\lambda}{f} - \varphi$ nec non $\varphi = ang.\,fin.\frac{+a\,fin.\,\lambda}{f} - \beta$.

Coroll. IX.

§. 283. Si anguli A et D ponantur fimul fumti uni recto vel etiam tribus rectis aequales, fit $f = \mp a\,fin.\lambda : cos.(\beta+\lambda)$; fed $tang.\,\lambda = \frac{f\,cos.\,\beta}{a \mp f\,fin.\,\beta}$ vel etiam $tang.\,\lambda = \frac{\mp f\,cot.\,\beta}{a\,cofec.\,\beta \mp f}$ atque

$$fin.\,\beta = \frac{\pm a\,fin.\,\lambda^2 \mp cos.\,\lambda\,\sqrt{(f^2 - a^2\,fin.\,\lambda^2)}}{f}.$$

Si anguli A et B ponantur fimul uni recto vel tribus rectis aequales fit, $f = -a\,fin.\,\lambda : cos.\,(\beta+\varphi)$

$$fin.\,\beta = \frac{\pm a\,fin.\,\lambda\,fin.\,\varphi \mp cos.\,\varphi\,\sqrt{(f^2 - a^2\,fin.\,\lambda^2)}}{f}$$

$$fin.\,\varphi = \frac{\pm a\,fin.\,\lambda\,fin.\,\beta \mp cos.\,\beta\,\sqrt{(f^2 - a^2\,fin.\,\lambda^2)}}{f}$$

Si anguli diagonaliter oppofiti B, D ponantur fimul fumti aequales uni recto vel tribus rectis, fit: $f = \mp a\,fin.\,\lambda : cos.\,(\beta + \psi)$ atque

$$fin.\,\beta = \frac{\pm a\,fin.\,\lambda\,fin\,\varphi \mp cos.\,\psi\,\sqrt{(f^2 - a^2\,fin.\,\lambda^2)}}{f}$$

$$fin.\,\psi = \frac{\pm a\,fin.\,\lambda\,fin.\,\beta \mp cos.\,\beta\,\sqrt{(f^2 - a^2\,fin.\,\lambda^2)}}{f}$$

Fit autem in cafu primo pofitione aequalitatis cum

cum uno recto, angulus $\beta = ang. \, ſin. \frac{a \, ſin. \lambda}{f} - \lambda$. Sed poſitione aequalitatis cum tribus rectis, $\beta = ang. \, ſin. \frac{+ a \, ſin. \lambda}{f} - \lambda$; per ſecundam poſitionem fit angulus $\beta = ang. \, ſin. \frac{- a \, ſin. \lambda}{f} - \varphi$ in caſu priori, ſed in altero erit $\beta = ang. \, ſin. \frac{+ ſin. \lambda}{f} - \varphi$ et angulus $\varphi = ang. \, ſin. \frac{- a \, ſin. d}{f} - \beta$, vel etiam in caſu altero $\varphi = ang. \, ſin. \frac{+ a \, ſin. \lambda}{f} - \beta$, per poſitionem tertiam fit $\beta = ang. \, ſin. \frac{- a \, ſin. \lambda}{f} - \psi$, vel in caſu etiam altero $\beta = ang. \, ſin. \frac{+ a \, ſin. d}{f} - \psi$, et angulus $\varphi = ang. \, ſin. \frac{- a \, ſin. \lambda}{f} - \beta$, vel etiam $\psi = ang. \, ſin. \frac{+ a \, ſin. d}{f} - \beta$.

Coroll. X.

§. 284. Si tres anguli A, B, D ſimul ſumti tribus rectis vel etiam uni recto aequales ponantur, ita ut in caſu poſteriore ſequatur angulum C fieri tribus rectis aequalem, cadente vertice C infra diagonalem BD et prodeunte trapezio inverſo fit, $f = \frac{\mp a \, ſin. \lambda}{cos. \beta}$, et $ſin. \beta = \frac{\sqrt{(f^2 - a^2 \, ſin. \lambda^2)}}{f}$

Si

Si anguli A, D, ACD ſint ſimul ſumti tribus rectis aequales, fit $f = a\ tang.\ \lambda$ et $tang.\ \lambda = \frac{f}{a}$.
Denique ſi anguli B, D, et ACD ponantur ſimul uni recto vel tribus rectis aequales, fit: $f = \pm a\ fin.\ \lambda\ cos\ \psi$, nec non $fin.\ \psi = \frac{\sqrt{(f^2 - a^2 fin.\ d^2}}{f}$.

Scholion III.

§. 285. Si trapezium ita ſit partialiter inverſum, ut totum triangulum ADC cadat intra triangulum $A'BC$, prodit haec aequatio: $a\ fin.\ \lambda = f\ fin.\ (\beta + \varphi - \lambda - \psi)$. Eadem aequatio prodit, ſi trapezium ſit totaliter inverſum, cadente latere AD intra latus AB, ſed latere CD extra latus CB. Verum ſi latus AD cadat extra AB, ſive latus CD ſit etiam extra vel intra latus CB, habetur haec aequatio: $a\ fin.\ \lambda = f\ fin.\ (\beta + \varphi + \psi - \lambda)$. Igitur generalis aequatio hujus problematis pro omni trapezio tam directo quam inverſo eſt hujusmodi: $a\ fin.\ \lambda = \mp f\ fin.\ (\beta + \varphi \pm \psi \pm \lambda)$.



CAPUT

CAPUT VI.

Continens ſex problemata quartae claſſis particularis, ſub priore principali contentae.

Problema XX.

Fig. XX. §. 286. In figura quadrilatera propoſita $ABCD$, inter omnia latera $AB=a$, $BC=b$, $AD=c$, $DC=d$, et diagonalem $AC=f$, et angulum combinationis $A=\psi$, aequationem invenire.

Solutio.

1) Ab angulo C et D in latus AB, demiſſis perpendiculis Cm, Da, et a vertice D, cum latere AB acta parallela, uſque ad Cm, erunt per principia Geometriae elementaris rectae Cm et Da inter ſe parallelae, et praeterea Dc aequalis ipſi am, et normalis ad Cm, et conſequenter triangulum DcC rectangulum.

2) Cum ſit per (Eucl. II. 11.) $AC^2=AB^2+BC^2-2AB.\ mB=f^2=a^2+b^2-2ab\ cos.\ B$; erit $cos.\ B=\frac{a^2+b^2-f^2}{2ab}$ et hinc $Bm=\frac{a^2+b^2-f^2}{2a}$, et conſequenter perpendiculum $Cm=$ $=\sqrt{\left(\frac{b^2-(a^2+b^2-f^2)^2}{4a^2}\right)}$, ſed eſt etiam perpendiculum $aD=c\ ſin.\psi$ et ſegmentum $Aa=c\ cos.\psi$, hinc

hinc obtinetur $Cc = Cm - mc = Cm - aD$
$= \sqrt{\left(\frac{b^2 - (a^2+b^2-f^2)^2}{4a^2}\right)} - c\ \mathit{sin.}\ \psi$, et
$am = Dc = \frac{a^2+f^2-b^2-2ac\ \mathit{cos.}\ \psi}{2a}$. Quare cum triangulum DcC sit rectangulum, ad aequationem pervenio per Theorema Pythagoricum hoc modo: $\left(\sqrt{\frac{(b^2-(a^2+b^2-f^2)^2)}{2a^2}} - c\ \mathit{sin.}\ \psi\right)^2$
$+ \frac{(a^2+f^2-b^2-2ac\ \mathit{cos.}\ \psi)^2}{4a^2} = d^2$. Ex qua sumtis semel quadratis numeratorum, in utroque termino sinistri membri, et sublatis terminis se mutuo destruentibus, et facta concinnatione obtinetur haec aequatio: $(b^2-a^2-f^2)c\ \mathit{cos.}\ \psi + (f^2+c^2-d^2)$
$a = c\ \mathit{sin.}\ \psi \sqrt{(4a^2b^2-(a^2+b^2-f^2)^2)}$. Q. E. I.

Coroll. I.

§. 287. Sumtis rursus quadratis habetur: $(b^2-a^2-f^2)^2 c^2 \mathit{cos.}\ \psi^2 + (f^2+c^2-d^2)^2 a^2 (b^2-a^2-f^2)$ $(f^2+c^2-d^2)\, 2ac\ \mathit{cos.}\ \psi = (2a^2b^2+2a^2f^2+2b^2f^2- a^4-b^4-f^4)\, c^2\ \mathit{sin.}\ \psi^2$; facta actuali evolutione terminorum propter $\mathit{sin.}\ \psi^2 + \mathit{cos.}\ \psi^2 = 1$, et identitatem signorum in quinque terminis, ortis ex quadratura termini primi sinistri membri, et quinque dextri membri ad partem sinistram translatis et aequatione ad nihilum reducta, habebitur: $(a^4+b^4+f^4-2a^2b^2-2b^2f^2)c^2 + 2a^2c^2f^2\ \mathit{cos.}\ 2\psi +$ $(c^4+f^4+d^4+2c^2f^2-2d^2f^2-2c^2d^2)a^2 - (f^4-c^2f^2+ d^2f^2+b^2f^2+b^2c^2-b^2d^2-a^2f^2-a^2c^2+a^2d^2)$ $2ac\ \mathit{cos.}\ \psi = 0$. Facto examine erit summa

 omni-

omnium coefficientium ipsius $f^4 = a^2 + c^2 - 2ac\ cos.\ \psi$, quae brevitatis ergo dicatur h; sed aggregatum omnium coefficientium ipsius f^2, erit $= a^2 c^2 (1 + cos. 2\psi) - 2b^2c^2 - 2a^2d^2 + (d^2 + b^2 - a^2 - c^2)$ $2\,a\,c\ cos.\ \psi$, quod brevitatis gratia dicatur $2M$; summa vero omnium terminorum in quibus non continetur f, erit $= (a^2-b^2)^2 c^2 + (c^2-d^2)^2 a^2 - (a^2-b^2)$ $(c^2-d^2)\ 2ac\ cos.\psi$, quae brevitatis gratia dicatur N; unde factis substitutionibus, erit aequatio abbreviata: $f^4 h + 2Mf^2 + N^2 = 0$, ex qua obtinetur

$$f = \frac{\surd(-M \pm \surd(M^2 - Nh))}{\surd h}$$

Coroll. II.

§. 288. Ex aequatione prouti ad eam primo est perventum longitudo lateris DC hoc modo exprimitur: $d = \frac{\surd(a^2+f^2-b^2-2ac\,cos.\psi)^2 +}{2a}$ $\frac{\surd(4a^2b^2-(a^2+b^2-f^2)^2-2ac\,fin.\psi)^2}{2a}$; ex eadem aequatione reducta finali apparet esse

$$c^2 + c\left(\frac{(b^2-a^2-f^2)\ cos.\ \psi - fin.\ \psi}{a}\right.$$
$$\left.\frac{\surd(4a^2b^2-(a^2+b^2-f^2)^2)}{a}\right) = d^2 - f^2.$$

Quod si ergo ad abbreviandum ponatur

$$\frac{(b^2-a^2-f^2)cos.\psi - fin.\psi\surd(4a^2b^2-(a^2+b^2-f^2)^2)}{a} = 2A$$

erit aequatio abbreviata $c^2 + 2Ac = d^2 - f^2$, unde elicitur $c = -A \pm \surd(A^2 + d^2 - f^2)$.

Coroll.

Coroll. III.

§. 289. Sit ad abbreviandum ex Corollario I. ſumma coefficientium quadrati ipſius $b = 2B$, et ſumma omnium terminorum abſolutorum $= Q$, erit haec aequatio abbreviata: $b^4c + 2Pb^2 + Q = 0$, ex qua extracta radice fit, $b = \frac{\sqrt{(-P \pm \sqrt{(P^2 - c^2 Q)})}}{c}$

Si facta diviſione totius aequationis ex Coroll. I. ad nihilum redactae per c^2 ponatur ad abbreviandum coefficiens ipſius $a^3, = A$, coefficiens ipſius $a^2 = B$, coefficiens ipſius $a, = C$, et ſumma omnium terminorum abſolutorum $= D$; habetur pro inveniendo latere AB haec aequatio abbreviata: $a^4 + Aa^3 + Ba^2 + Ca + D = 0$.

Coroll. IV.

§. 290. Sit ad abbreviandum $(\sqrt{(4a^2b^2 - (a^2+b^2-f^2)^2)}) = P$; $(b^2-a^2-f^2)c = Q$; $(f^2+c^2-d^2)a = R$ factis ſubſtitutionibus et tranſpoſitione, erit aequatio abbreviata: $Q\sqrt{(1-x^2)} = Px - R$, poſito $\mathit{ſin.}\ \psi = x$, et $\cos.\ \psi = \sqrt{(1-x^2)}$, ex qua ſumtis quadratis et facta reductione, obtinetur haec aequatio: $x = \frac{PR \pm Q\sqrt{(P^2+Q^2-R^2)}}{P^2+Q^2} = \mathit{ſin.}\psi$.

Scholion I.

§. 291. Inter ſex caſus qui in aequatione continentur nullus datur Tetragonometriae proprius; ſed omnes utrique methodo et trigonometricae et tetragonometricae pariter ſubſunt, et quidem ita ut trigonometrice naturalius et facilius expediantur, tamen omnes ſex caſus evolvi, quia ex aequatione haud difficulter ſequntur.

N 2 *Coroll.*

Coroll. V.

§. 292. Si angulus A ponatur aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus et trapezio in triangulum BCD abeunte, aequatio, quae ex resolutione prodiit, in hanc mutatur: $(f^2+c^2-d^2)(a-(b^2-a^2-f^2)c=o$, unde sequitur fore $f=\frac{\surd((b^2-a^2)c+(d^2-c^2)a)}{a+c}$ vel etiam $f=\frac{\surd((b+a)(b-a)c+(d+c)(d-c)a)}{a+c}$; ex eadem elicitur $d=\frac{\surd((c^2+f^2)a+(a^2+f^2-b^2)c)}{a}$; ex eadem sequitur etiam $b=\frac{\surd((a^2+f^2)c+(f^2+c^2-d^2)a)}{c}$, porro ex eadem erit $a=\frac{d^2-f^2-c^2\pm\surd((f^2+c^2-d^2)^2-4c(b^2-f^2))}{2c}$, nec non $c=\frac{b^2-f^2-a^2\pm\surd((a^2+f^2-b^2)^2-4a^2(d^2-f^2))}{2a}$.

Scholion II.

§. 293. Harum aequationum secunda et quinta ex aequationibus generalibus in corollario secundo inventis etiam consequntur, tres autem caeterae, quod mireris, ex suis generalibus in Corollariis primo et tertio non consequuntur, cujus phaenomeni ratio est, quod aequatio initio Corollarii primi sumtis quadratis et facta evolutione et transpositione terminorum in aliam formam fuit transfusa, ex qua legitimi valores litterarum f, a, b, non consequntur, nisi prius ad for-

formam priſtinam reducatur; idem enim ex qualibet ejusdem aequationis forma non ſemper legitime ſequitur. Sed hac methodo nova quadam ratione ſolvitur problema ad triangula pertinens, quod ordinaria methodo reſolvendum foret hoc modo: Demiſſo ab angulo C in baſin BD perpendiculo Ca, erit per (Eucl. II. 11.) $b^2 = a^2 + f^2 - 2af\ cos.\ A$, et $d^2 = c^2 + f^2 + 2cf\ cos.\ A$ et ad eliminandum $cos.\ A$ aequationibus combinatis habetur: $\frac{b^2 - a^2 - f^2}{c} = \frac{c^2 + f^2 - d^2}{c}$, unde conſequitur fore uti prius: $f = \frac{\surd((b^2 - a^2)c + (d^2 - c)a)}{a + c}$.

Coroll. VI.

§. 294. Si in aequatione in primo Corollario inventa ponatur angulus A rectus, vel etiam tribus rectis aequalis, cadente vertice A ſupra diagonalem BD intra vel extra triangulum BCD, et trapezio inverſo prodeunte, habetur pro diagonali haec aequatio: $f = \frac{\surd((a^2 d^2 + b^2 c^2 \pm}{\surd(a + c)}$

$$\frac{\surd(a^2 d^2 + b^2 c^2)^2 - ((c^2 - d^2)a^2 + c^2(a^2 - b^2)(a^2 + c^2)))}{\surd(a + c)}$$

ubi in caſu anguli A recto aequalis ſignum affirmativum, in caſu altero, quo tribus rectis aequalis, ſignum negativum valere debet, ſed ex corollario II. fit $d = \frac{\surd((a^2 + f^2 - b^2)^2 -}{2a}$

$$\frac{(\surd(4a^2 b^2 - (a^2 + b^2 - f^2)^2) - 2ac)^2)}{2a}$$

 $c =$

$$c = \frac{\sqrt{(4a^2b^2 - (a^2+b^2-f^2)^2)} \pm \sqrt{(4a^2b^2 - (a^2+b^2-f^2)^2 + 4a^2(d^2-f^2))}}{2a}$$

Quia in aequatione quarti gradus in corollario III. vi hujus pofitionis fit $A=o$, et $C=o$, ut aequatio in hanc mutetur: $a^4+Ba^2+D=o$; extracta radice invenitur effe $a = \frac{\sqrt{(-B \pm \sqrt{(B^2-4D)}}}{\sqrt{2}}$

et in aequationis $b = \frac{\sqrt{(-P \pm \sqrt{(P^2-c^2Q)}}}{c}$

termino P evanefcit fumma omnium terminorum, quae efficitur $cos.\ \psi$.

Scholion III.

§. 295. Aequatio hujus Problematis primo inventa nullas recipit varietates, nifi quae ex ambiguitate figni radicalis et cofinus oriuntur, quare pro omni trapezio tam directo, quam inverfo primo inventa valet aequatio.

Problema XXI.

Fig. XXI. §. 296. In figura quadrilatera propofita $ABCD$; inter haec fex, latus $AB=a$, $BC=b$, $AD=d$, diagonalem $AC=f$, et duos angulos $A=\psi$, $D=\varphi$; aequationem invenire.

Solutio.

1) In triangulo dextro formo hanc analogiam $AC(f)$: *fin.* D (*fin.* φ) $= AD\ (c)$: *fin.* $ACD = \frac{c\ fin.\ \varphi}{f}$ et hinc $cos. ACD = \frac{\sqrt{(f^2 - c^2 fin.\ \varphi^2)}}{f}$.

2) Ex

2) Ex his et ex ſinu coſinuque anguli D, formo ſinum et coſinum anguli tertii CAD et invenio

$$\mathit{fin.}\, CAD = \frac{\mathit{fin.}\,\varphi \sqrt{(f^2 - c^2 \mathit{fin.}\,\varphi^2)} + c\,\mathit{fin.}\,\varphi\, \mathit{cos.}\,\varphi}{f}$$

nec non $\mathit{cos.}\ CAD =$

$$= \frac{-\mathit{cos.}\,\varphi \sqrt{(f^2 - c^2\, \mathit{fin.}\,\varphi^2)} + c\,\mathit{fin.}\,\varphi^2}{f}.$$

3) Ex his et ex ſinu coſinuque anguli A formo coſinum anguli CAB, et invenio eſſe: $\mathit{cos.}\,(\psi - CAD) =$

$$= \frac{-\mathit{cos.}\,\psi\, \mathit{cos.}\,\varphi \sqrt{(f^2 - c^2\, \mathit{fin.}\,\varphi^2)} + \mathit{cos.}\,\psi\, \mathit{fin.}\,\varphi^2}{f}$$

$$+ \frac{\mathit{fin.}\,\psi\, \mathit{fin.}\,\varphi \sqrt{(f^2 - c^2 \mathit{fin.}\,\varphi^2)} + c\,\mathit{fin.}\,\psi\, \mathit{fin.}\,\varphi\, \mathit{cos.}\,\varphi}{f}$$

$$= \frac{-\mathit{cos.}\,\varphi + \psi) \sqrt{(f^2 - c^2 \mathit{fin.}\,\varphi^2)} + c\,\mathit{fin.}\,\varphi\, \mathit{fin.}\,(\varphi + \psi)}{f}$$

$= \mathit{cos.}\ CAB.$

4) His factis in triangulo ſiniſtro per tritum theorema (Eucl. II. 11.) ad hanc aequationem pervenio: $BC^2 = AB^2 + AC^2 - 2\,Ab\ AC$, et ſymbolis ſubſtitutis $b^2 = a^2 + f^2 + 2a\,\mathit{cos.}\,(\varphi + \psi) \sqrt{(f^2 - c^2 \mathit{fin.}\,\varphi)} - 2ac\,\mathit{fin.}\,\varphi\, \mathit{fin.}\,(\varphi + \psi)$ Q. E. I.

Coroll. I.

§. 297. Ex hac aequatione ſtatim incurrit in oculos, eſſe pro inveniendo latere BC, $b = \sqrt{(a^2 + f^2 + 2a\,\mathit{cos.}\,(\varphi + \psi) \sqrt{(f^2 - c^2 \mathit{fin.}\,\varphi^2)} - 2ac\,\mathit{fin.}\,\varphi\ \mathit{fin.}\,(\varphi + \psi))}$ Sit ad abbreviandum $\mathit{fin.}\,\varphi = h$, $\mathit{fin.}\,(\varphi + \psi) = m$, $\mathit{cos.}\,(\varphi + \psi) = n$, erit pro inveniendo latere AD, ſubſtitutione et

 trans-

transpositione facta $b^2-a^2-f^2+2achm=2an\sqrt{(f^2-c^2h^2)}$, ex qua operatione absoluta obtinetur haec aequatio:

$$c=\frac{(a^2+f^2-b^2)m\pm n\sqrt{(4a^2f^2-(b^2-a^2-f^2)^2)}}{2ah}$$

Coroll. II.

§. 298. Sed facta substitutione et transpositione pro obtinendo latere AB, habetur haec aequatio abbreviata: $a+2a(n\sqrt{(f^2-c^2h^2)}-chm)=b^2-f^2$, et operatione ad finem perducta obtinetur haec: $a=chm-n\sqrt{(f^2-c^2h^2)}\pm\sqrt{((n\sqrt{(f^2-c^2h^2)}-chm)^2+b^2-f^2)}$. Verum pro diagonali obtinenda habetur: $f=\sqrt{(b^2+a^2(n^2-m^2)+2achm\pm 2an\sqrt{(b^2-(am-ch)^2)})}$ et si ad abbreviandum ponatur $n^2-m^2=cos.(\phi+\psi)-fin.(\phi+\psi)^2=cos.2(\phi+\psi)=p$, habetur: $f=(\sqrt{b^2+a^2p+2achm\pm 2an\sqrt{(b^2-(am-ch)^2)}})$.

Coroll. III.

§. 299. Pro obtinendo angulo D, ex aequatione inventa elicio hanc aequationem: $2ac\,fin.\,\phi=-A^2\,fin.(\phi+\psi)\pm cos.(\phi+\psi)\sqrt{(4a^2f^2-A^4)}$ in qua brevitatis gratia positum erat, $b^2-a^2-f^2=A^2$; ponatur rursus ad abbreviandum $\sqrt{(4a^2f^2-A^4)}=B^2$ et ex aequatione abbreviata: $2ac\,fin.\,\phi=-A^2 fin.(\phi+\psi)+B^2 cos.(\phi+\psi)$ elicitur $tang.\,\phi=(B^2 cos.\,\psi-A^2 fin.\,\psi):(2ac+A^2 cos.\,\psi+B^2 fin.\,\psi)$ et si ad abbreviandum ponatur $2ac\,fin.\,\phi=h$, invenitur esse:

fin.

$$ſin.(\varphi+\psi)=\frac{-A^2 h^2 \pm B^2 \surd(A^4-B^4-h^4)}{A^4+B^4}$$

et conſequenter ipſe angulus

$$\psi=ang.ſin.\frac{(-A^2 h^2 \pm B^2 \surd(A^4+B^4-h^4))}{A^4+B^4}-\varphi$$

Scholion I.

§. 300. Nec in hoc problemate datur ullus caſus Tetragonometriae proprius, ſed omnes utrique methodo et trigonometricae et tetragonometricae pariter ſubſunt, ſed ita ut fere facilius et expeditius trigonometrice ſolvantur, excepto caſu, quo diagonalis quaeritur, pro quo aequatio exhibita brevitate ſolutioni trigonometricae parum aut nihil cedit; omnes autem caſus evolvi, cum ex aequatione haud difficulter et operoſe ſequantur.

Coroll. IV.

§. 301. Si anguli A et D ponantur ſimul ſumti duobus rectis aequales, ut prodeat trapezium parallelarum baſium AB, CD; aequatio, quae prius prodiit, in hanc mutatur: $b^2=a^2+f^2-2a\surd(f^2-c^2 ſin.\varphi^2)$, vel etiam $b^2=a^2+f^2-2a\surd(f^2-c^2 ſin.\psi^2)$, unde extracta radice fit, $f=\surd(a^2+b^2 \pm 2a\surd(b^2-c^2 ſin.\varphi^2))$; id quod etiam ex ipſa aequatione generali pro diagonali inventa elicitur.

Coroll. V.

§. 302. Si ponantur anguli A et D, ſimul ſumti uni recto vel tribus rectis aequales; aequa-

aequatio pro diagonali in hanc mutatur: $f = \sqrt{(b^2 - a^2 \pm 2ac\, sin.\, \phi)}$, ubi signum superius valet pro positione anguli recti, inferius pro positione trium rectorum.

Coroll. VI.

§. 303. Si in aequatione generali pro diagonali ponatur angulus A aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus, et quadrilatero in triangulum abeunte, aequatio in hanc mutatur: $f = \sqrt{(b^2 + a^2 p - 2ach^2 + 2an\sqrt{(b^2 - h^2(a+c)^2)})}$, in qua est $cos.\, \phi^2 - sin.\, \phi^2 = cos.\, 2\phi = p$, $m == - h$, $n == - cos.\, \phi$.

Scholion II.

§. 304. Solvit haec aequatio nova quadam ratione problema de invenienda recta EC, ab angulo C ad latus BD ducta, quod ordinaria methodo per duas analogias trigonometricas solvendum esset. Similiter pro segmentis BE, DE et latere BC e suis valoribus generalibus novae solutiones eliciuntur sequentes.

Coroll. VII.

§ 305. Pro latere BC inveniendo habetur: $b = \sqrt{(a^2 + f^2 - 2a\, cos.\, \phi \sqrt{(f^2 - c^2 sin.\, \phi^2)} + 2ac\, sin.\, \phi^2)}$ pro segmento DE haec habetur:

$$e = \frac{(b^2 - a^2 - f^2)\, sin.\, \phi^2 \mp cos.\, \phi \sqrt{(4a^2 f^2 - (b^2 - a^2 - f^2)^2)}}{2a\, sin.\, \phi},$$

denique pro segmento altero BE haec invenitur: $a == cos.\, \phi \sqrt{(f^2 - c^2 sin.\, \phi^2)} - c\, sin.\, \phi^2 \pm \sqrt{((c\, sin.\, \phi^2 - cos.\, \phi \sqrt{(f^2 - c^2 sin.\, \phi^2)}) + b^2 - f^2)}$

et

et pro angulo D, $\text{tang.}\,\phi = -B^2 : (A^2 - 2ac)$. Quod ſi ponatur angulus D aequalis duobus rectis, lateribus AD, DC in diagonalem incidentibus et trapezio in triangulum ABC abeunte, aequatio pro diagonali in hanc abit $f = (b^2 + a^2 p \mp 2an\sqrt{(b^2 - a^2 m^2)})$, et iterum extracta radice $f = an \pm \sqrt{(b^2 - a^2 m^2)}$, in qua eſt $m = -\sin.\psi$, $n = -\cos.\psi$, $\cos.\psi^2 - \sin.\psi^2 = \cos 2\psi = p$; fit autem in hoc caſu $\sin.\psi = B^2 : \sqrt{(A^4 + B^4)}$.

Coroll. VIII.

§. 306. Si angulus A ponatur aequalis recto vel tribus rectis, cadente vertice A ſupra diagonalem BD intra vel extra triangulum BCD, et trapezio inverſo prodeunte, fit aequatio pro diagonali, $f = (b^2 - a^2 p \pm 2achm \mp 2an\sqrt{(b^2 - (am \mp ch)^2)})$, in qua aequatione ſigna ſuperiora valent pro poſitione recti denotante $m = \pm \cos.\phi$, $n = \mp \sin.\phi$, $\sin.\phi^2 - \cos.\phi^2 = -\cos.2\phi = m^2 - n^2 = -p$, in hac hypotheſi invenitur fore latus $BC = b$ $= \sqrt{(a^2 + f^2 + 2a\sin.\phi\sqrt{(f^2 - c^2\sin.\phi^2)} \mp 2ac\sin.\phi\cos.\phi)}$

$$AD = c = \frac{\pm(a^2 + f^2 - b^2)m \pm n\sqrt{(4a^2 f^2 - (b^2 - a^2 - f^2)^2)}}{2ah}$$

et denique latus $AB = a = \pm chm \mp n\sqrt{(f^2 - c^2 h^2)} \pm \sqrt{((\mp n\sqrt{(f^2 - c^2 h^2)} \mp chm)^2 + b^2 - f^2)}$ et $\text{tang.}\,\phi = \mp A^2 : (2ac \pm B^2)$, in his quatuor aequationibus ſigna valent, ut in aequatione pro diagonali.

Coroll. IX.

§. 307. Si in aequatione pro diagonali ponatur angulus D rectus vel tribus rectis aequalis, cadente vertice D a ſiniſtra diagonalis intra vel

extra

extra triangulum ABC, et trapezio inverſo prodeunte, prodit aequatio plane ſimilis priori in qua eſt $m = \pm cos.\psi$, $n = \mp fin.\psi$, $fin.\psi^2 - cos.\psi^2 = m^2 - n^2 = -p = -cos.2\psi$, fit autem in hac hypotheſi, latus $BC = b = \surd(a^2 + f^2 \mp 2a fin.\psi \surd(f^2 - c^2) \mp 2ac cos.\psi)$ latus vero $AD = c = \frac{\pm(a + f^2 - b^2)m \pm n \surd(4a^2f^2 - (b^2 - a^2 - f^2)^2)}{2a}$;

in hac aequatione ſigna ſuperiora valent pro poſitione trium rectorum, inferiora pro poſitione recti. Latus $AB = a = cm \mp n \surd(f^2 - c^2) \pm \surd((f^2 - c^2)n^2 - cm)^2 + b^2 - f^2)$.

Coroll. X.

§. 308. Quod ſi ponatur $am = ch$, ut ſit $a : fin.\phi = c : fin.(\phi + \psi)$, aequatio pro diagonali in hanc mutatur: $f = \surd(b^2 + a^2p + 2achm \pm 2abn)$, quae am et ch pro ſe mutuo, et pro p ſuum valorem ſubſtituendo, in hanc ſimpliciorem abit, $f = \surd((b \pm an)^2 + ch)$. Si vero ponatur $b = am - ch$, fit $f = \surd(a^2n^2 + c^2h^2)$ pro b et p ſuis valoribus ſubſtitutis, vel etiam pro ch rurſus ſuo valore ſurrogato fit $f = \surd(a^2 + b^2 - 2abm)$.

Scholion III.

§. 309. Aequatio in ſolutione hujus problematis data pro trapezio directo, valet in hypotheſi figurae conſtructae, quando ſumma angulorum D et ACD excedit rectum, verum quando minor eſt recto et angulus D major angulo A, habetur haec aequatio: $b^2 = a^2 + f^2 - 2a cos.(\phi - \psi) \surd(f^2 - c^2 fin.\phi^2) + 2ac fin.\phi fin.(\phi - \psi)$, ſed angu-

angulo A, angulum D excedente haec: $b^2 = a^2 + f^2 - 2a \cos.(\psi - \varphi) \sqrt{(f^2 - c^2 \sin.\varphi^2)} - 2ac \sin.\varphi \sin.(\psi - \varphi)$. Quare generalis aequatio pro omni trapezio directo ſit hujusmodi: $b^2 = a^2 + f^2 \pm 2a \cos.\left(\begin{matrix}\psi \pm \varphi \\ \varphi - \psi\end{matrix}\right) \sqrt{(f^2 - c^2 \sin.\varphi^2)} \mp 2ac \sin.\left(\begin{matrix}\psi \pm \varphi \\ \varphi - \psi\end{matrix}\right) \sin.\varphi$. Quod ſi trapezium ſit partialiter inverſum, ut totum triangulum ADC cadat intra triangulum ABC, exiſtente ſumma angulorum D et ACD recto majore, angulo D majore quam angulo A, habetur haec aequatio: $b^2 = a^2 + f^2 + 2a \cos.(\varphi - \psi) \sqrt{(f^2 - c^2 \sin.\varphi^2)} - 2ac \sin.\varphi \sin.(\varphi - \psi)$; verum caeteris manentibus iisdem, exiſtente angulo A majore angulo D, habetur: $b^2 = a^2 + f^2 + 2a \cos.(\psi - \varphi) \sqrt{(f^2 - c^2 \sin.\varphi^2)} + 2ac \sin\varphi \sin.(\psi - \varphi)$; ſed, exiſtente ſumma angulorum D et ACD recto minore, habetur haec aequatio: $b^2 = a^2 + f^2 - 2a \cos.(\varphi + \psi) \sqrt{(f^2 - c^2 \sin.\varphi^2)} + 2ac \sin.\varphi \sin.(\varphi + \psi)$. Caeterum ex aequatione liquet plures variationes occurrere non poſſe, quomodocunque figura variet, et aequatio generaliſſima erit pro omni trapezio, quae jam allata fuit pro trapezio directo.

Problema XXII.

§. 310. In figura quadrilatera propoſita $ABCD$ inter haec ſex, latus $AB = a$, $BC = b$, $CD = d$, diagonalem $AC = f$, angulum $A = \psi$, et $D = \varphi$ aequationem invenire. **Fig. XXII.**

Solutio.

Solutio.

1) In triangulo dextro formo hanc analogiam: $AC(f):\, fin.\, D\, (fin.\, \phi) = CD\, (d):\, fin.\, CAD = \frac{d\, fin.\, \phi}{f}$ et confequenter $cos.\, CAD = \frac{\surd(f^2 - d^2\, fin.\, \phi^2)}{f}$

2) Ex his et ex finu cofinuque anguli A formo cofinum anguli CAB, et invenio effe: $cos.\, (\psi - CAD) = \frac{cos.\, \psi \surd(f^2 - d^2 fin.\, \phi^2) + d\, fin.\, \phi\, fin.\, \psi}{f}$

3) His factis in triangulo finiftro pervenio ad hanc aequationem per (Eucl. II. 11.) $BC^2 = AB^2 + AC^2 - 2\, AB.\, AC$ et fubftitutis fymbolis habetur: $b^2 = a^2 + f^2 - 2\, a\, cos.\, \psi \surd(f^2 - d^2 fin.\, \phi^2) - 2\, a\, d\, fin.\, \phi\, fin.\, \psi$, Q. E. I.

Coroll. 1.

§. 311. Ex hac aequatione flatim liquet effe: latus $BC = b = \surd(a^2 + f^2 - 2\, a\, cos.\, \psi \surd(-f^2 - d^2 fin.\, \phi^2) - 2\, a\, d\, fin.\, \phi\, fin.\, \psi)$ et pofito ad abbreviandum $fin.\, \phi = h$, $fin.\, \psi = m$, $cos.\, \psi = n$, invenitur effe latus $AB = a = n \surd(f^2 - d^2 h^2) + d\, h\, m \pm \surd((n \surd(f^2 - d^2 h^2) + d\, h\, m)^2 + b^2 - f^2)$ et latus $DC = d = \frac{(a^2 + f^2 - b^2)\, m \pm n \surd(4 a^2 f^2 - (a^2 + f^2 - b^2)^2)}{2\, a\, n}$, atque diagonalis $AC = f = \surd(b^2 + a^2 p + 2\, a\, d\, h\, m \pm 2\, a\, n \surd(b^2 - (a\, m - d\, h)^2))$, in qua aequatione eft $p = n^2 - m^2 = cos.\, \psi^2 - fin.\, \psi^2 = cos.\, 2\, \psi$; pro angulo D inveniendo hanc aequationem confequor: $fin.\, \phi = \frac{m(a^2 + f^2 - b^2) \pm n \surd(4 a^2 f^2 - (a^2 + f^2 - b^2)^2)}{2\, a\, d}$

deni-

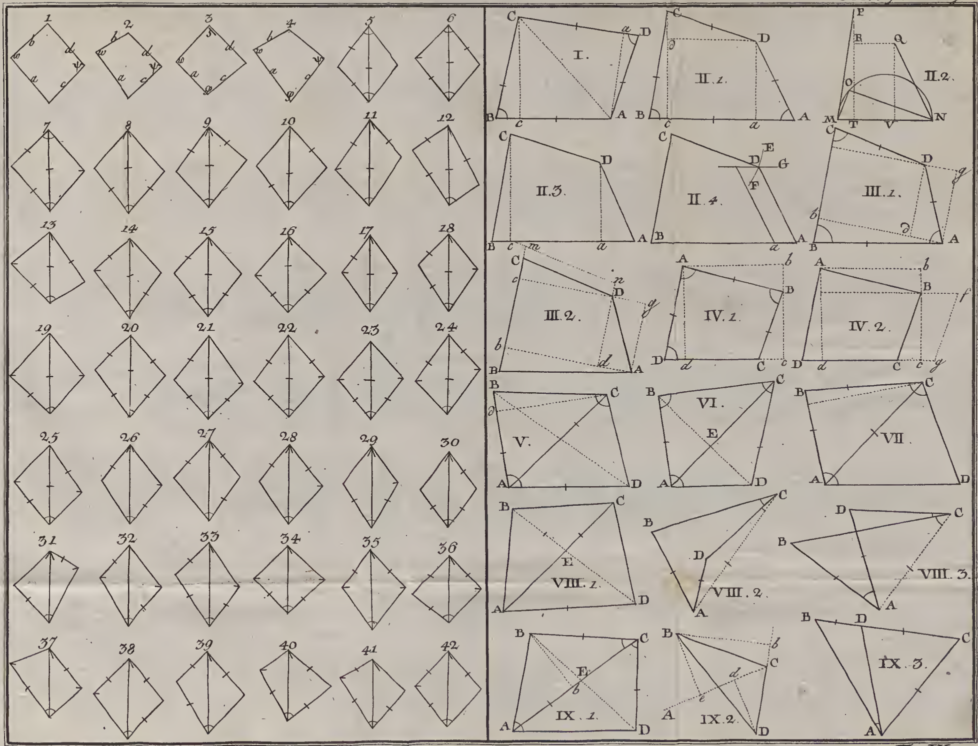

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
I.
II. 1.
II. 2.
II. 3.
II. 4.
III. 1.
III. 2.
IV. 1.
IV. 2.
V.
VI.
VII.
VIII. 1.
VIII. 2.
VIII. 3.
IX. 1.
IX. 2.
IX. 3.

Fig. IX.

denique pro angulo A habetur haec aequatio:

$$\text{ſin.}\,\psi = \frac{d\,h\,(a^2 + f^2 - b^2) \pm \surd\,(f^2 - d^2\,h^2)}{2\,a\,f^2}$$
$$\frac{(2\,a^2 f^2 + 2\,b^2 f^2 + 2\,a^2 b^2 - a^4 - b^4 - f^4))}{2\,a\,f^2}.$$

Scholion I.

§. 312. Inter ſex caſus qui in aequatione continentur, non datur niſi unus Tetragonometriae proprius, qui tum obtinet quando diagonalis quaeritur, ſuppeditat autem hic caſus novum in Geometria practica problema ſatis elegans pulchrum et utile, quod ita ſe habet: In figura quadrilatera ex datis tribus lateribus et duobus angulis lateri ignoto adjacentibus, invenire diagonalem et conſtruere figuram Caeteri quinque caſus utrique methodo, et trigonometricae et tetragonometricae pariter ſubſunt, ſed ita ut trigonometrice facilius expediantur, omnes tamen caſus evolvi, quia ex aequatione non adeo prolixe et operoſe eruuntur, et aequationibus haud inconcinnis exprimuntur.

Coroll. II.

§. 313. Si anguli A et D, ponantur ſimul ſumti aequales duobus rectis, ut prodeat trapezium parallelarum baſium $A\,B$, $D\,C$ fit aequatio pro diagonali: $f = \surd\,(b^2 + a^2\,p + 2\,a\,d\,h \pm : a\,n\,\surd\,(b^2 - h^2\,(a\,c)^2))$; fit autem in hoc caſu: latus $BC = b = \surd\,(a^2 + f^2 - 2\,a\,n\,\surd\,(f^2 - d^2 h^2) - 2\,a\,d\,h^2)$ et latus $A\,B = a = n\,\surd\,(f^2 - d^2\,h^2) + d\,h^2 \pm \surd\,((n\,\surd\,(f^2 - d^2\,h^2) + d\,h^2)^2 + b^2 - f^2)$ et

latus

latus $DC = d =$

$$= \frac{a^2+f^2-b^2 \pm cot.\psi \sqrt{(4a^2f^2-(a^2+f^2-b^2)^2)}}{2a\, cot.\psi.}$$

Coroll. III.

§. 314. Si angulus A duobus rectis aequalis ponatur, lateribus AB, AD, cum diagonali BD coincidentibus, et trapezio in triangulum BCD abeunte; prodit pro diagonali sive recta ab angulo C, in latus sibi oppositum BD ducta, haec aequatio: $f = \sqrt{(a^2+b^2 \pm 2a\sqrt{(b^2-d^2h^2)})}$ fit autem in hoc casu $b = \sqrt{(a^2+f^2 \pm \sqrt{(f^2-d^2 fin.\varphi^2)})}$ $a = \sqrt{(b^2-d^2h^2)} \pm \sqrt{(f^2-d^2h^2)}$, et

$$d = \frac{\sqrt{(4a^2-f^2-(a^2+f^2-b^2)^2)}}{2ah}$$

$$fin.\varphi = \frac{\sqrt{(4a^2f^2-(a^2+f^2-b^2)^2)}}{2ad}.$$

Scholion II.

§. 315. His igitur aequationibus nova quadam ratione solvuntur triangula tum partialia BED, et ECD, tum etiam totale BCD, quae methodis vulgaribus trigonometricis perparum aut nihil ceduut.

Coroll. IV.

§. 316. Si angulus A ponatur uni recto vel tribus rectis aequalis, cadente vertice A supra diagonalem BD, intra vel extra triangulum BCD et trapezio inverso prodeunte, prodit pro diagonali haec aequatio $f = \sqrt{(a^2+b^2 \pm 2adh)}$

pro

pro latere BC habetur haec: $b = \surd(a^2 + f^2 \mp 2adh)$, pro latere AB obtinetur: $a = \pm dh \pm \surd(b^2 - f^2 + d^2h^2)$ denique pro latere DC eſt $d = \frac{a^2 + f^2 - b^2}{2ah}$ et pro angulo habetur *ſin.* $\varphi = \frac{a^2 + f^2 - b^2}{2ad}$.

Scholion III.

§. 317. In tribus aequationibus prioribus valent ſigna ſuperiora pro poſitione anguli recti; inferiora pro poſitione trium rectorum, ſed in duabus ultimis expreſſiones in utraque poſitione recidunt eodem, quia angulus D, qui in priore caſu in figura conſtructa eſt affirmativus, in poſitione trium rectorum fit negativus, hoc eſt ad dextram lateris CD cadit, et trapezium fit totaliter inverſum.

Coroll. V.

§. 318. Quod ſi angulus D ponatur rectus, vel etiam tribus rectis aequalis, cadente vertice D, ad ſiniſtram diagonalis intra vel extra triangulum ABC et denuo prodeunte trapezio inverſo, prodit pro diagonali haec aequatio: $f = \surd(b^2 + a^2p \pm 2adm \pm 2an\surd(b^2 - (am \mp c)^2))$ ſed in hoc caſu fit: $b = \surd(a^2 + f^2 - 2an\surd(f^2 - d^2) \mp 2adm)$, et latus $AB = a = n\surd(f^2 - d^2) \pm dm \pm \surd(n\surd(f^2 - d^2) \pm dm)^2 + b - f^2)$ et pro angulo A habetur haec aequatio: *ſin.* $\psi = \frac{\pm d(a^2 + f^2 - b^2)}{2af}$ $\frac{\pm \surd((f^2 - d^2)\ (2a^2f^2 + 2b^2f^2 + 2a^2b^2 - a^4 - b^4 - f^4))}{2af}$

O ubi

ubi ſignum ſuperius valet pro angulo recto, inferius pro tribus rectis.

Coroll. VI.

§. 319. Si ponatur $am = dh$, ut ſit $a : fin.\,\phi = d : fin.\,\psi$, erit $f = \surd((b^2 \pm an)^2 + d^2h^2)$, ſi vero ponatur $b = am - dh$, fit $f = \surd(a^2n^2 + d^2h^2)$, vel etiam $f = \surd(a^2 + b^2 - 2abm)$; his addere licet, ſi ponatur $f = d$, fore ex aequatione, quae primo prodiit: $b^2 = a^2 + f^2 - 2ad\,cos.(\phi\,cos.\,\psi + fin.\,\phi\,fin.\,\psi) = a^2 + f^2 - 2ad(\psi - \phi)$ vel etiam fiet $b^2 = a^2 + f^2 - 2ad\,cos.(\phi - \psi)$, denique fieri poteſt $b^2 = a^2 + f^2 + 2ad\,(cos.\,\phi\,cos.\,\psi - fin.\,\phi\,fin.\,\psi) = a^2 + f^2 + 2ad\,cos.(\phi + \psi)$.

Scholion IV.

§. 320. Primus caſus obtinet in figura propoſita quando latus AD verſus dextram ab angulo A cadit et conſequenter quando angulus D vel huic aequalis CAD minor eſt angulo A. Secundus caſus obtinet, quando latus AD cadit verſus ſiniſtram ab angulo A, et quidem extra latus AB, ut prodeat trapezium totaliter inverſum, in hoc enim caſu angulus $CAD = D$, fit major angulo A. Denique tertius caſus obtinet, quando latus AD uti prius cadit ad ſiniſtrum anguli D, ſed intra latus AB, ut trapezium prodeat partialiter inverſum, in hoc enim caſu angulus CAB eſt ſumma angulorum A et D.

Scholion. V.

§. 321. Aequatio hujus problematis ſimplicior praecedente, nullas recipit varietates, niſi quod utrumque ſignum praeponendum ſit termino

mino tertio et quarto dextri aequationis membri, quare generalis aequatio pro omni trapezio tam directo, quam inverso, est hoc modo exprimenda $b^2 = a^2 + f^2 \mp 2\,a\,cos.\,\psi \surd (f^2 - d^2 fin.\,\varphi^2) \mp 2\,a\,d\,fin.\,\varphi\,fin.\,\psi$.

Problema XXIII.

§. 322. In figura quadrilatera $ABCD$ propofita inter haec fex: latus $AB=a$, $AD=c$, diagonalem $AC=f$, et tres angulos $A=\psi$, $B=\lambda$, $D=\varphi$, aequationem invenire. Fig. XXIII.

Solutio.

1) Cum dentur nomina primitiva angulorum A, B, D; erit nomen derivativum fummae eorundem $\lambda+\varphi+\psi$ et hinc $fin.\,(A+B+D) = fin.\,(\lambda+\varphi+\psi)$ et $cos.\,(A+B+D) = cos.\,(\lambda+\varphi+\psi)$, quare cum in folutione problematis XXI. num. 1. (§. 296.) fit inventus

$$fin.\,ACD = \frac{c\,fin.\,\varphi}{f} \text{ et } cos.\,ACD = \frac{\surd(f^2 - c^2 fin.\,\varphi^2)}{f};$$

ex his et praecedentibus formo finum fmmmae angulorum omnium A, B, D, ACD, et invenio effe $fin.\,(A+B+D+ACD) =$

$$= \frac{c\,fin.\,\varphi\,cos.\,(\lambda+\varphi+\psi) + fin.\,(\lambda+\varphi+\psi)\surd(f^2 - c^2 fin.\,\psi^2)}{f}$$

2) Jam ob formatum finum fummae omnium angulorum in quadrilatero, excepto angulo ACB, et fummam omnium quatuor rectis aequalem $= 2\,\pi$; erit nomen derivativum finus anguli

$$ACB = \text{fin.}\,(2\pi - (A+B+D+ACD = \frac{-\text{fin.}\,(\lambda+\phi+\psi)\,\surd(f^2 - c^2\,\text{fin.}\,\phi^2) -}{f}$$

$$\frac{-c\,\text{fin.}\,\phi\,\text{cof.}\,(\lambda+\phi+\psi)}{f} = \text{fin.}\,ACB.$$

3) His peractis in triangulo finistro pervenio ad hanc analogiam $AC:\text{fin.}\,B = AB:\text{fin.}\,ACB$ et consequenter symbolis substitutis habetur:

$$f:\text{fin.}\,\lambda = a\frac{-\text{fin.}\,(\lambda+\phi+\psi)\,\surd(f^2-c^2\,\text{fin.}\,\phi^2)}{f}$$

$$\frac{+c\,\text{fin.}\,\phi\,\text{cof.}\,(\lambda+\phi+\psi)}{f} \text{ atque}$$

$$a\,\text{fin.}\,\lambda = -\text{fin.}\,(\lambda+\phi+\psi)\,\surd(f^2-c^2\,\text{fin.}\,\phi^2) - c\,\text{fin.}\,\phi\,\text{cof.}\,(\lambda+\phi+\psi).$$ Q. E I.

Scholion I.

§. 323. Cum in triangulo finistro fit $\text{fin.}\,ACB = \frac{a\,\text{fin.}\,\lambda}{f}$, et $\text{cof.}\,ACB = \frac{\surd(f^2-a^2\,\text{fin.}\,\lambda^2)}{f}$ et hinc $\text{fin.}\,(A+B+D+ACB) =$

$$= \frac{a\,\text{fin.}\,\lambda\,\text{cof.}\,(\lambda+\phi+\psi) + \text{fin.}\,(\lambda+\phi+\psi)}{f}$$

$$\frac{\surd(f^2 - a^2\,\text{fin.}\,\lambda^2)}{f},$$ per eandem ac prius rationem sequitur fore: $\text{fin.}\,ACD = \frac{-a\,\text{fin.}\,\lambda\,\text{cof.}\,(\lambda+\phi+\psi)}{f}$

$$\frac{-\text{fin.}\,(\lambda+\phi+\psi)\,\surd(f^2 - a^2\,\text{fin.}\,\lambda^2)}{f}$$ et in triangulo dextro hac analogia formata: $AC:\text{fin.}\,D = AD:\text{fin.}\,ACD$ ad hanc aequationem per-

pervenio, $c\,fin.\,\phi = -fin.(\lambda+\phi+\psi)\sqrt{(f^2-a^2 fin.\lambda^2)} - a\,fin.\lambda\,cof.(\lambda+\phi+\psi)$, haec autem eſt priori ſimilis et ex illa derivatur a et c, λ et ϕ ſibi mutuo ſubſtituendo.

Scholion II.

§. 224. Cum per ipſum formationis modum ſinus angulorum ACB, et ACD negativi prodierint, neceſſario inde ſequebatur totum dextrum aequationis membrum fieri negativum, id quod primo ſane intuitu abſurdi quid involvere videtur; haec autem abſurdi apparentia evaneſcit, ſi conſideretur terminum priorem dextri membri ſigno radicali affectum eſſe, et per conſequens tam affirmative, quam negative ſumi poſſe, unde ſequitur ex negativo in affirmativum mutari poſſe; id quod nova ratione confirmatur, ſi attendatur unam aequationem eſſe alterius radicem negativam; nam ſi ex priori aequatione per extractionem radicis quadraticae, quaeras valorem $c\,fin.\,\phi$, obtinebis hanc aequationem $c\,fin.\,\phi = -a\,fin.\,\lambda\,cof.\,(\lambda+\phi+\psi) \pm fin.\,(\lambda+\phi+\psi)\sqrt{(f^2-a^2 fin.\,\lambda^2)}$ unde liquet aequationem poſteriorem eſſe radicem negativam prioris; quod ſi vero ex poſteriore quaeras valorem $a\,fin.\,\lambda$, ad hanc aequationem pervenies: $a\,fin.\,d = -c\,fin.\,\phi\,cof.\,(\lambda+\phi+\psi) \pm fin.\,(\lambda+\phi+\psi)\sqrt{(f^2-c^2 fin.\,\phi^2)}$, quare liquet priorem eſſe poſterioris radicem negativam, igitur utraque aequatio ſigno affirmativo adjecto generaliter repraeſentatur, aequationes ergo non realiter, ſed formaliter differunt.

 Coroll.

Coroll. I.

§. 325. Ex aequatione priori ſtatim ſequitur eſſe $a = \frac{\pm c \, \text{fin.}\, \phi \, \text{cos.}\, (\lambda+\phi+\psi) \pm \text{fin.}\, (\lambda+\phi+\psi)}{\text{fin.}\, \lambda}$ $\frac{\surd (f^2 - c^2 \, \text{fin.}\, \phi^2)}{\text{fin.}\, \lambda}$ quod idem radicem extrahendo ex poſteriore obtinetur. Sed ex aequatione poſteriore ſtatim liquet eſſe:

$$c = \frac{\pm a \, \text{fin.}\, \lambda \, \text{cos.}\, (\lambda+\phi+\psi) \pm \text{fin.}\, (\lambda+\phi+\psi)}{\text{fin.}\, \phi}$$
$$\frac{\surd (f^2 - a^2 \, \text{fin.}\, d^2)}{\text{fin.}\, \phi}$$

quod idem ex priore radicem extrahendo obtinetur; ſed pro diagonali obtinetur ex utraque aequatione haec aequatio:

$$f = \frac{\surd (c^2 \, \text{fin.}\, \phi^2 \pm 2ac \, \text{fin.}\, \lambda \, \text{fin.}\, \phi \, \text{cos.}\, (\lambda+\phi+\psi) + a^2 \, \text{fin.}\, d^2)}{\text{fin.}\, (\lambda+\phi+\psi)}$$

vel etiam haec ad conſtruendum aptior:

$$f = \frac{\surd (a^2 \, \text{fin.}\, \lambda^2 \, \text{fin.}\, (\lambda+\phi+\psi)^2 + (c \, \text{fin.}\, \phi \pm}{\text{fin.}\, (\lambda+\phi+\psi)}$$
$$\frac{a \, \text{fin.}\, \lambda \, \text{cos.}\, (\lambda+\phi+\psi))^2)}{\text{fin.}\, (\lambda+\phi+\psi)}.$$

Coroll. II.

326. Pro angulo B obtinendo habetur haec aequatio: $\text{tang.}\, \lambda =$

$$= \frac{\pm c \, \text{fin.}\, \phi \, \text{cos.}\, (\phi+\psi) \pm \text{fin.}\, (\phi+\psi) \surd (f^2 - c^2 \, \text{fin.}\, \phi^2)}{a + c \, \text{fin.}\, \phi \, \text{fin.}\, (\phi+\psi) \mp \text{cos.}\, (\phi+\psi) \surd (f^2 - c^2 \, \text{fin.}\, \phi^2)}$$

vel etiam $\text{tang.}\, \lambda =$

$$= \frac{c \, \text{fin.}\, \phi \pm \text{tang.}\, (\phi+\psi) \surd (f^2 - c^2 \, \text{fin.}\, \phi^2)}{a \, \text{fec.}\, (\phi+\psi) \pm c \, \text{fin.}\, \phi \, \text{tang.}\, (\phi+\psi) \mp \surd (f^2 - c^2 \, \text{fin.}\, \phi^2)}$$

ſimili-

ſimiliter pro angulo D habetur haec aequatio:
$tang.\ \varphi = \frac{\pm a\,sin.\lambda\,cos.(\lambda+\psi) \pm sin.(\lambda+\psi)\surd(f^2 - a^2\,sin\,\lambda^2)}{c + a\,sin.\lambda\,sin.(\lambda+\psi) \mp sin.(\lambda+\psi)\surd(f^2 - a^2\,sin.\lambda^2)}$;
vel etiam haec: $tang.\ \varphi = \frac{a\,sin.\lambda \pm tang.(\lambda+\psi)\surd(f^2 - a^2\,sin.\lambda^2)}{c\,sec.(\lambda+\psi) \pm a\,sin\,\lambda\,tang.(\lambda+\psi) \mp \surd(f^2 - a^2\,sin.\lambda^2)}$.
Denique multiplicando aequationem priorem per $c\,sin.\varphi$, et poſteriorem per $a\,sin.d$, et aequationes inter ſe connectendo habetur pro angulo A haec aequatio: $tang.\ (\lambda+\varphi+\psi) = \frac{(a\,sin.\lambda + c\,sin.\varphi)\surd(a\,sin.\lambda - c\,sin.\varphi)}{c\,sin.\varphi\surd(f^2 - c^2\,sin.\varphi^2) \mp a\,sin.d\surd(f^2 - a^2\,sin\,\lambda^2)}$
et a tangente ad angulum ipſum deſcendendo habetur: $\psi = ang.\ tang.\ \frac{(a\,sin.\lambda + c\,sin.\varphi)(a\,sin.d - c\,sin.\varphi)}{c\,sin.\varphi\surd(f^2 - c^2\,sin.\varphi^2) \mp a\,sin.d\surd(f^2 - a^2\,sin.d^2)} - \lambda - \varphi$.

Scholion III.

§. 327. Hic monendum, quemadmodum fieri poteſt ut $a\,sin.\lambda$ majus $c\,sin.\varphi$, ita etiam contingere poſſe, ut ſit minus, quo in caſu aequatio rectius ſcribitur hoc modo;
$tang.\ (\lambda+\varphi+\psi) = \frac{(a\,sin.\lambda + c\,sin.\varphi)(c\,sin.\varphi - a\,sin.\lambda)}{c\,sin.\varphi\surd(f^2 - c^2\,sin.\varphi^2) \pm a\,sin.\lambda\surd(f^2 - a^2\,sin.\lambda^2)}$
et conſequenter etiam Angulus $A = \psi = ang.\ tang.$
$\frac{(a\,sin\,\lambda + c\,sin.\varphi)(c\,sin.\varphi - a\,sin.\lambda)}{c\,sin.\varphi\surd(f^2 - c^2\,sin.\varphi^2) \pm a\,sin.\lambda\surd(f^2 - a^2\,sin.\lambda^2)} - \lambda - \varphi$.

Scholion IV.

§. 328. Inter ſex caſus hujus problematis nullus datur Tetragonometriae proprius, ſed omnes utrique methodo et trigonometricae et tetragonometricae pariter ſubſunt. Trium vero caſuum ſolutiones, quibus duo latera AB, CD, et diagonalis quaeruntur, ſolutionibus trigonometricis parum vel etiam nihil cedunt. Sed anguli trigonometrice fere facilius quam tetragonometrice expediuntur, omnes tamen caſus evolvi, quia ex aequationibus faciliter et breviter ſequuntur.

Coroll. III.

§. 329. Si anguli A et D ſimul ſumti ſint duobus rectis aequales prodeunte ſic trapezio parallelarum baſium AB, DC, habetur pro diagonali haec aequatio:

$$f = \frac{\sqrt{(a^2 \, \text{ſin.}\, \lambda^4 + (c \, \text{ſin.}\, \phi \mp a \, \text{ſin.}\, \lambda \cos.\, \lambda)^2)}}{\text{ſin.}\, \lambda},$$

ſi vero angulus A et B ſimul ſumti ſint aequales duobus rectis, prodeunte ſic trapezio parallelarum baſium BC, AD, habetur pro diagonali haec aequatio:

$$f = \frac{\sqrt{(a^2 \, \text{ſin.}\, \lambda^2 \, \text{ſin.}\, \phi^2 + (c \, \text{ſin.}\, \phi \mp a \, \text{ſin.}\, \lambda \cos.\, \phi)^2)}}{\text{ſin.}\, \phi}$$

Denique ſi anguli diagonaliter oppoſiti ſint aequales duobus rectis, ut prodeat trapezium circulo inſcriptibile, habetur pro diagonali haec aequatio:

$$f = \frac{\sqrt{(a^2 \, \text{ſin.}\, \lambda^2 \, \text{ſin.}\, \psi^2 + (c \, \text{ſin.}\, \phi \mp a \, \text{ſin.}\, \lambda \cos.\, \psi)^2)}}{\text{ſin.}\, \psi}.$$

Coroll.

Coroll. IV.

§. 330. In prima hypotheſi habetur latus $AB = a = \frac{\pm c\, \text{ſin.}\, \phi \cos. \lambda \pm \text{ſin.}\, \lambda \sqrt{(f^2 - c^2 \text{ſin.}\, \phi^2}}{\text{ſin.}\, d} =$
$= \pm c\, \text{ſin.}\, \phi \cot. \lambda \pm \sqrt{(f^2 - c^2 \text{ſin.}\, \phi^2)}$; et in ſecunda hypotheſi fit,

$$a = \frac{\pm c\, \text{ſin.}\, \phi \cos. \phi \pm \text{ſin.}\, \phi \sqrt{(f^2 - c^2 \text{ſin.}\, \phi^2)}}{\text{ſin.}\, \lambda}$$

in tertia vero fit,

$$a = \frac{\pm c\, \text{ſin.}\, \phi \cos. \psi \pm \text{ſin.}\, \psi \sqrt{(f^2 - c^2 \text{ſin.}\, \phi^2}}{\text{ſin.}\, \lambda}$$

ſimiliter in prima hypotheſi fit latus $c = \pm a \cos. d \pm \sqrt{(f^2 - a^2 \text{ſin.}\, \lambda^2)}$, at in ſecunda

$$c = \frac{\pm a\, \text{ſin.}\, \lambda \cos. \psi \pm \text{ſin.}\, \psi \sqrt{(f^2 - a^2 \text{ſin.}\, \lambda^2)}}{\text{ſin.}\, \lambda}$$

at in tertia

$$c = \frac{\pm a\, \text{ſin.}\, \lambda \cos. \psi \pm \text{ſin.}\, \psi \sqrt{(f^2 - a^2 \text{ſin.}\, \lambda^2)}}{\text{ſin.}\, d},$$

ſed in prima hypotheſi fit $\text{tang.}\, \lambda = c\, \text{ſin.}\, \phi : (a \pm \sqrt{(f^2 - c^2 \text{ſin.}\, \phi^2)})$ in ſecunda hypotheſi fit $\text{tang.}\, \phi = a\, \text{ſin.}\, \lambda : (c \pm \sqrt{(f^2 - a^2 \text{ſin.}\, \lambda^2)})$. Denique in tertia hypotheſi haec aequatio habetur pro angulo $A = \psi = \text{ang. tang.}$

$$\frac{(a^2 - c^2)\, \text{ſin.}\, \lambda}{c \sqrt{(f^2 - c^2 \text{ſin.}\, \lambda^2)} \pm a \sqrt{(f^2 - a^2 \text{ſin.}\, \lambda^2)}},$$ vel etiam

$$\psi = \text{ang. tang.} \frac{(c^2 - a^2)\, \text{ſin.}\, \phi}{c \sqrt{(f^2 - c^2 \text{ſin.}\, \phi^2)} \pm a \sqrt{(f^2 - a^2 \text{ſin.}\, \phi^2)}}.$$

Coroll. V.

§. 331. Si anguli A et D ponantur simul sumti uni recto vel tribus rectis aequales, habetur pro diagonali haec aequatio:

$$f = \frac{\surd(a^2 \sin.\lambda^2 \cos.\lambda^2 + (c \sin.\phi \pm a \sin.\lambda^2)^2)}{\cos.\lambda}.$$

Si anguli A et B sint simul uni recto vel tribus rectis aequales, fit

$$f = \frac{\surd(a^2 \sin.\lambda^2 \cos.\phi^2 + (c \pm a \sin.\lambda)^2 \sin.\phi^2)}{\cos.\phi}$$

quae in hanc mutatur: $f = \surd(a^2 \sin.\lambda^2 + (c \pm a \sin.\lambda)^2 \mathit{tang.}\,\phi^2)$; si denique anguli B et D, diagonaliter oppositi sint uni recto vel tribus rectis aequales, habetur haec aequatio:

$$f = \frac{\surd(a^2 \sin.\lambda^2 \cos.\psi^2 + (c \sin.\phi \pm a \sin.\lambda \sin.\psi)^2)}{\cos.\psi}.$$

In primo casu fit aequatio pro latere AB $= a = \pm c \sin.\phi \pm \cot.\lambda \surd(f^2 - c^2 \sin.\phi^2)$, in secundo casu fit $a = \dfrac{\pm c \sin.\phi^2 \pm \cos.\phi \surd(f^2 - c^2 \sin.\phi^2)}{\sin.\lambda}$, in tertio casu fit

$$a = \frac{\pm c \sin.\phi \sin.\psi \pm \cos.\psi \surd(f^2 - c^2 \sin.\phi^2)}{\sin.\lambda}.$$

In primo casu fit aequatio pro latere AD $c = \dfrac{\pm a \sin.\lambda^2 \pm \cos.\lambda \surd(f^2 - a^2 \sin.\lambda^2)}{\sin.\phi}$, in secundo casu fit $c = \pm a \sin.\lambda \pm \cot.\phi \surd(f^2 - a^2 \sin.d^2)$, denique casu tertio

$$c = \frac{\pm a \sin.\lambda \sin.\psi \pm \cos.\psi \surd(f^2 - a^2 \sin.\lambda^2)}{\sin.\phi}.$$

Porro

Porro in primo caſu fit pro angulo B, $tang.\,\lambda = \surd(f^2 - c^2\,\text{ſin}.\,g^2) : (a \pm c\,\text{ſin}.\,\varphi)$, in ſecundo caſu fit pro angulo D, $tang.\,\varphi = \surd(f^2 - a^2\,\text{ſin}.\,\lambda^2) : (c \pm a\,\text{ſin}.\,\lambda)$, in tertio caſu fit pro angulo A obtinendo, $\psi = ang.\ cot.$ $\frac{(a\cos.\varphi + c\,\text{ſin}.\,\varphi)\ (a\cos.\varphi - c\,\text{ſin}.\,\varphi)}{c\,\text{ſin}.\,\varphi\surd(f^2 - c^2\,\text{ſin}.\,\varphi^2) \pm a\cos\varphi\surd(f^2 - a^2\cos.\varphi^2)}$.

Coroll. VI.

§. 332. Si angulus A ponatur rectus vel tribus rectis aequalis, cadente vertice A ſupra diagonalem intra vel extra triangulum BCD, et trapezio inverſo prodeunte, habetur pro diagonali haec aequatio: $f = \frac{\surd(a^2\,\text{ſin}.\,\lambda^2\cos.(\lambda+\varphi)^2}{\cos.(\lambda+\varphi)}$ $\frac{+ (c\,\text{ſin}.\,\varphi \pm a\,\text{ſin}.\,\lambda\,\text{ſin}.(\lambda+\varphi))^2)}{\cos.(\lambda+\varphi)}$. Si angulus D ponatur rectus vel tribus rectis aequalis cadente vertice D ad ſiniſtram diagonalis intra vel extra ſiniſtrum triangulum et trapezio inverſo prodeunte, habetur pro diagonali haec aequatio:

$$f = \frac{\surd(c^2 \mp 2ac\,\text{ſin}.\,\lambda\,\text{ſin}.(\lambda+\psi) + a^2\,\text{ſin}.\,\lambda^2)}{\cos.(\lambda+\psi)}.$$

Si angulus B ponatur rectus vel tribus rectis aequalis, cadente vertice B ad dextram diagonalis intra vel extra triangulum dextrum, et trapezio inverſo prodeunte, habetur pro diagonali haec aequatio:

$$f = \frac{\surd(a^2\cos.(\varphi+\psi)^2 + (c\,\text{ſin}.\,\varphi \pm a\,\text{ſin}.(\varphi+\psi)^2))}{\cos.(\varphi+\psi)}.$$

Coroll.

Coroll. VII.

§. 333. In prima hypothesi fit latus $AB =$

$$= a = \frac{\pm c\,\textit{fin}\,\phi\,\textit{fin}.(\lambda+\phi) \pm \cos.(\lambda+\phi)\,\surd(f^2 - c^2\,\textit{fin}.\phi^2)}{\textit{fin}.\,\lambda},$$

at in secunda fit

$$a = \frac{-c\,\textit{fin}.(\lambda+\psi) \pm \cos.(\lambda+\psi)\,\surd(f^2 - c^2)}{\textit{fin}.\,\lambda},$$ denique

in tertia, $f = \mp c\,\textit{fin}.\,\phi\,\textit{fin}.(\phi+\psi) \mp \cos.(\phi+\psi)$ $\surd(f^2 - c^2\,\textit{fin}.\phi^2)$; sed in primo casu habetur: latus $AD = c =$

$$= \frac{\pm a\,\textit{fin}.\lambda\,\textit{fin}.(\lambda+\phi) \pm \cos.(\lambda+\phi)\,\surd(f^2 - a^2\,\textit{fin}.\lambda^2)}{\textit{fin}.\,\phi}$$

et in secundo

$$c = \frac{\mp a\,\textit{fin}.\lambda\,\textit{fin}.(\lambda+\psi) \mp \cos.(\lambda+\psi)\,\surd(f^2 - a^2\,\textit{fin}.\lambda^2)}{\textit{fin}.\,\phi}$$

et in tertio

$$c = \frac{-a\,\textit{fin}.(\phi+\psi) \pm \cos.(\phi+\psi)\,\surd(f^2 - a^2)}{\textit{fin}.\,\phi}.$$

Coroll. VIII.

§. 334. In prima hypothesi fit pro angulo B

$$\textit{tang}.\,\lambda = \frac{\mp c\,\textit{fin}.\,\phi^2 \pm \cos.\phi\,\surd(f^2 - c^2\,\textit{fin}.\phi^2)}{a \pm c\,\textit{fin}.\,\phi\cos.\phi \pm \textit{fin}.\phi\,\surd(f^2 - c^2\,\textit{fin}.\phi^2)}$$

et in secunda $\textit{tang}.\,\lambda = \dfrac{-c\,\textit{fin}.\psi \pm \cos.\psi\,\surd(f^2 - c^2)}{a + c\cos.\psi \pm \textit{fin}.\psi\,\surd(f^2 - c^2)}$ in prima hypothesi habetur pro angulo D haec expressio:

$$\textit{tang}.\,\phi = \frac{\mp a\,\textit{fin}.\,\lambda^2 \pm \cos.\lambda\,\surd(f^2 - a^2\,\textit{fin}.\lambda^2)}{c \pm a\,\textit{fin}.\lambda\cos.\lambda \pm \textit{fin}.\lambda\,\surd(f^2 - a^2\,\textit{fin}.\lambda^2)},$$

sed

fed in tertia fit $tang.\,\varphi = \frac{-a\,fin.\,\psi \pm cos.\,\psi \surd (f^2 - a^2)}{c + a\,cos.\,\psi \pm \surd (f^2 - a^2)}$; in fecunda hypothefi habetur pro angulo A haec expreffio:

$$cot.\,(\lambda + \psi) = \frac{(a\,fin.\,\lambda + c)\,(a\,fin.\,\lambda - c)}{\pm c \surd (f^2 - c^2) \pm a\,fin.\,\lambda \surd (f^2 - a^2 fin.\,\lambda^2)},$$

et angulus ipfe

$$\psi = ang.\,cot.\frac{(a\,fin.\,\lambda + c)\,(a\,fin.\,\lambda - c) - \lambda}{\mp c \surd (f^2 - c^2) \pm a\,fin.\,\lambda \surd (f^2 - a^2 fin.\,\lambda^2)}.$$

Sed in tertia

$$cot.\,(\varphi + \psi) = \frac{(c\,fin.\,\varphi + a)\,(c\,fin.\,\varphi - a)}{\mp c\,fin.\,\varphi \surd (f^2 - c^2 fin.\,\varphi^2) \pm a \surd (f^2 - a^2)}$$

et angulus ipfe

$$\psi = ang.\,cot.\frac{(c\,fin.\,\varphi + a)\,(c\,fin.\,\varphi - a) - \varphi}{\mp fin.\,\varphi \surd (f^2 - c^2 fin.\,\varphi^2) \pm a \surd (f^2 - a^2)}.$$

Coroll. IX.

§. 335. Si ponatur angulus A aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus et trapezio in triangulum BCD abeunte, habetur pro diagonali haec expreffio:

$$f = \frac{\surd (a^2 fin\lambda^2 fin\,(\lambda + \varphi)^2 + (c\,fin\varphi \pm a\,fin\lambda\,cos(\lambda + \varphi^2))^2)}{cos.\,(\lambda + \varphi)},$$

in hac autem hypothefi fit fegmentum,

$$EB = a = \frac{\pm\, c\,fin.\,\varphi\,cos.\,(\lambda + \varphi) \mp fin.\,(\lambda + \varphi)}{fin.\,\lambda}$$

$\frac{\surd (f^2 - c^2\,fin.\,\varphi^2)}{fin.\,\lambda}$ fed alterum fegmentum ED

fit $c = \frac{\pm a\,fin.\,\lambda\,cos.(\lambda + \varphi) \mp fin.(\lambda + \varphi) \surd (f^2 - a^2 fin.\,\lambda^2)}{fin.\,\varphi}$

pro

pro angulo B habetur haec expreſſio:

$$tang.\,\lambda = \frac{fin.\,\varphi\,(\pm c\,cos.\,\varphi \mp \surd(f^2 - c^2 fin.\,\varphi^2))}{a - c\,fin.\,\varphi^2 \pm cos.\,\varphi\,\surd(f^2 - c^2 fin.\,\varphi^2)},$$

et pro angulo D

$$tang.\,\varphi = \frac{fin.\,\lambda\,(a\,cos.\,\lambda \mp \surd(f^2 - a^2 fin.\,\lambda^2))}{c - a\,fin.\,\lambda^2 \pm cos.\,\lambda\,\surd(f^2 - a^2 fin.\,\lambda^2)}.$$

Denique ponendo angulo D vel B duobus rectis aequali, prodit in caſu priore $a\,fin.\,\lambda = f\,fin.(\lambda+\psi)$, et in poſteriore $c\,fin.\,\varphi = f\,fin.\,(\varphi+\psi)$, unde reſultant conſuetae analogiae trigonometricae una in triangulo ABC, $f : fin.\,\lambda = a : fin.(\lambda+\psi)$, in triangulo vero ACD haec altera, $f : fin.\,\varphi = c : fin.(\varphi+\psi)$, quo ipſo aequationes et operationes verificantur:

Scholion V.

§. 336. In triangulo BCD totali et partialibus BCE, CDE expreſſiones pro recta CE et ſegmentis BE, DE, nec non angulis B, D novas quasdam ſolutiones ſuppeditant, quae ſolutionibus trigonometricis parum cedunt.

Coroll. X.

§. 337. Si in aequatione priore ponatur $f = c$, aequatio mutatur in hanc adhibito ſigno radicali negativo et prioris termino negativo: $a\,fin.\,\lambda = -c\,fin.\,(2\varphi+\lambda+\psi)$ ſed affirmativo adhibito et prioris etiamnum negativo: $a\,fin.\lambda = c\,fin.\,(\lambda+\psi)$, quae locum non habet niſi angulus D, fiat aequalis duobus rectis et trapezium abeat in triangulum ſiniſtrum ABC, ſed in poſteriore aequatione poſito $f = a$, et adhibitis ſignis negativis habetur:

c fin.

$c\,fin.\,\varphi = -a\,fin.\,(2\lambda + \varphi + \psi)$, fed affirmativo et prioris termini negativo adhibito haec: $c\,fin.\,\varphi = a\,fin.\,(\varphi + \psi)$, quae locum non habet nifi angulus B fit aequalis duobus rectis, et trapezium abeat in triangulum ADC.

Scholion VI.

§. 338. In aequatione hujus problematis pro trapezio directo nullae varietates occurrunt, nifi quae ex ambiguitate figni radicalis et cofinus oriuntur. Quod fi trapezium fit partialiter inverfum ita ut totum triangulum ADC cadat intra triangulum ABC habetur haec aequatio: $a\,fin.\,\lambda = fin.\,(\varphi - \lambda - \psi)\,\sqrt{(f^2 - c^2 fin.\,\varphi^2)} + c\,fin.\,\varphi\,cos.\,(\varphi - \lambda - \psi)$. Si trapezium ita fit totaliter inverfum ut latus AD cadat intra latus AB, CD vero extra CB; prodit eadem aequatio cum fignis contrariis in dextro membro. Sin autem trapezium ita fit totaliter inverfum ut latus AD cadat extra latus AB, CD vero intra CB habetur haec aequatio: $a\,fin.\,\lambda = fin.\,(\varphi + \psi - \lambda)\,\sqrt{(f^2 - c^2 fin.\,\varphi^2)} + c\,fin.\,\varphi\,cos.\,(\varphi + \psi - \lambda)$. Denique fi trapezium ita fit partialiter inverfum ut totum triangulum ABC cadat intra triangulum ADC, prodit eadem aequatio, cum figno negativo termini, qui irrationalitate eft adfectus, quare generalis aequatio pro omni trapezio tam directo quam inverfo, hanc habet formam: $a\,fin.\,\lambda = \mp fin.\,(\varphi \pm \psi \pm \lambda)\,\sqrt{(f^2 - c^2 fin.\,\varphi^2)} \mp c\,fin.\,\varphi\,cos.\,(\varphi \pm \psi \pm \lambda)$.

Proble-

Problema XXIV.

Fig. XXIV. §. 339. In figura quadrilatera proposita $ABCD$ inter haec ſex; latus $BC=b$, $DC=d$, diagonalem $AC=f$, et tres angulos $A=\psi$, $B=\lambda$, et $D=\varphi$, aequationem invenire.

Solutio.

1) Ex ſolutione problematis XXII. (§. 310.) habetur $\text{ſin.}\ CAD=\frac{d\ \text{ſin.}\ \varphi}{f}$ et $\text{cos.}\ CAD=$
$=\frac{\sqrt{(f^2-d^2\ \text{ſin.}\ \varphi^2)}}{f}$, ex quibus et ex ſinu coſinuque anguli A ſequitur eſſe $\text{ſin.}\ (\psi - CAD)=$
$=\frac{\text{ſin.}\ \psi\sqrt{(f^2-d^2\ \text{ſin.}\ \varphi^2)}-d\ \text{ſin.}\ \varphi\ \text{cos.}\ \psi}{f}=\text{ſin.}\ CAB.$

2) Jam in triangulo ſiniſtro pervenio ad hanc analogiam $AC:\ \text{ſin.}\ B=BC:\ \text{ſin.}\ CAB$, h. e.
$f:\text{ſin.}\ \lambda=b:\frac{\text{ſin.}\ \psi\sqrt{(f^2-d^2\ \text{ſin.}\ \varphi^2)}-d\ \text{ſin.}\ \varphi\ \text{cos.}\ \psi}{f}$
et conſequenter habetur: $b\ \text{ſin.}\ \lambda = \text{ſin.}\ \psi$
$\sqrt{(f^2-d^2\ \text{ſin.}\ \varphi^2)}-d\ \text{ſin.}\ \varphi\ \text{cos.}\ \psi$, Q. E. I.

Scholion I.

§. 340. Similiter cum ſit in triangulo ſiniſtro $\text{ſin.}\ CAB=\frac{b\ \text{ſin.}\ \lambda}{f}$ et $\text{cos.}\ CAB=\frac{\sqrt{(f^2-b^2\ \text{ſin.}\ \lambda^2)}}{f}$
ex his et ex ſinu coſinuque anguli A uti prius formando ſinum anguli CAD, invenio ipſum eſſe:
$\frac{\text{ſin.}\ \psi\sqrt{(f^2-b^2\ \text{ſin.}\ \lambda^2)}-b\ \text{cos.}\ \psi\ \text{ſin.}\ \lambda}{f}=\text{ſin.}\ CAD,$
et conſequenter ad hanc aequationem priori ſimilem

lem pervenio: $d\,\text{fin.}\,\phi = \text{fin.}\,\psi \sqrt{(f^2 - b^2\,\text{fin.}\,\lambda^2)} - b\,\text{fin.}\,\lambda \cos\psi$; haec autem ex illa priori prodit, fi in ipfa $b\,\text{fin.}\,\lambda$ et $d\,\text{fin.}\,\phi$ loca fua permutent. Quod fi vero ex aequatione priori quaeratur valor $d\,\text{fin.}\,\phi$; invenitur effe, $d\,\text{fin.}\,\phi = - b\,\text{fin.}\,\lambda \cos.\psi \pm \text{fin.}\,\psi \sqrt{(f^2 - b^2\,\text{fin.}\,\lambda^2)}$ et vice verfa, fi ex pofteriore quaeratur valor $b\,\text{fin.}\,d$ invenitur effe, $b\,\text{fin.}\,\lambda = - d\,\text{fin.}\,\phi \cos.\psi \pm \text{fin.}\,\phi \sqrt{(f^2 - d^2\,\text{fin.}\,\phi^2)}$; una igitur aequatio eft alterius radix adhibito figno radicali affirmativo, confequenter aequationes non realiter fed formaliter tantum differunt.

Coroll. I.

§. 341. Ex aequatione priore ftatim liquet effe

$$b = \frac{\text{fin.}\,\psi \sqrt{(f^2 - d^2\,\text{fin.}\,\phi^2)} \pm d\,\text{fin.}\,\phi \cos.\psi}{\text{fin.}\,\lambda},$$

fed idem radicem extrahendo elicitur ex pofteriore; fed ex hac ftatim liquet effe,

$$d = \frac{\text{fin.}\,\psi \sqrt{(f^2 - b^2\,\text{fin.}\,\lambda^2)} \pm b\,\text{fin.}\,\lambda \cos.\psi}{\text{fin.}\,\phi},$$

pro diagonali vero ex utraque hic idem valor elicitur

$$f = \frac{\sqrt{(b^2\text{fin.}\,\lambda^2 \pm 2\,b\,d\,\text{fin.}\,\lambda\,\text{fin.}\,\phi \cos.\psi + d^2\text{fin.}\,\phi^2}}{\text{fin.}\,\psi}$$

vel etiam hic alter

$$f = \frac{\sqrt{(b^2\text{fin.}\,\lambda^2\text{fin.}\,\psi^2 + (d\,\text{fin.}\,\phi \pm b\,\text{fin.}\,\lambda \cos\psi)^2)}}{\text{fin.}\,\psi}.$$

Coroll. II.

§. 342. Similiter ex aequatione priori ftatim incurrit in oculos effe:

$$\text{fin.}\,\lambda = \frac{\text{fin.}\,\psi \sqrt{(f^2 - d^2\,\text{fin.}\,\phi^2)} \pm d\,\text{fin.}\,\phi \cos.\psi}{b}$$

 et

et hinc ipſum angulum

$$\lambda = ang.\,fin.\frac{(fin.\psi\sqrt{(f^2-d^2 fin.\phi^2)}\pm d\,fin.\phi\cos.\psi)}{b};$$

et ex aequatione poſteriore eſſe

$$fin.\,\phi = \frac{fin.\,\psi\sqrt{(f^2-b^2 fin.\lambda^2)}\pm b\,fin.\lambda\cos.\psi}{d}$$

et ipſum angulum

$$\phi = ang.\,fin.\frac{(fin.\psi\sqrt{(f^2-b^2 fin.\lambda^2)}\pm b\,fin.\lambda\cos.\psi)}{d}$$

Denique aequationibus ita inter ſe combinatis, ut prior per $d\,fin.\,\phi$, poſterior per $b\,fin.\,\lambda$ multiplicetur, pro angulo A haec aequatio obtinetur:

$$tang.\,\psi = \frac{(d\,fin.\,\phi + b\,fin.\,\lambda)\,(d\,fin.\,\phi - b\,fin.\,\lambda)}{d\,fin.\phi\sqrt{(f^2-d^2 fin.\phi^2)}\pm b\,fin.\lambda\sqrt{(f^2-b^2 fin.\lambda^2)}}$$

et ipſe angulus $\psi = ang.\,tang.$

$$\frac{(d\,fin.\,\phi + b\,fin.\,\lambda)\,(d\,fin.\,\phi - b\,fin.\,\lambda)}{d\,fin.\,\phi\sqrt{(f^2-d^2 fin.\,\phi^2)}\pm b\,fin.\,\lambda\sqrt{(f^2-b^2 fin.\,\lambda^2)}}.$$

Scholion II.

§. 343. Quemadmodum fieri poteſt, ut ſit $d\,fin.\,\phi$ majus $b\,fin.\,d$, ita etiam contrarium evenire poteſt, quo in caſu ſcribendum erit,

$$tang.\,\psi = \frac{(d\,fin.\,\phi + b\,fin.\,\lambda)\,(b\,fin.\lambda - d\,fin.\phi)}{d\,fin.\phi\sqrt{(f^2-d^2 fin.\phi^2)}\pm b\,fin.\lambda\sqrt{(f^2-b^2 fin.\lambda^2)}}.$$

Verum inter ſex caſus in aequatione comprehenſos nullus datur Tetragonometriae proprius, ſed omnes utrique methodo et trigonometricae et tetragonometricae pariter ſubſunt, ſolutiones tamen quibus diagonalis et latera inveniuntur, trigonometricis parum vel etiam nihil cedunt. Nec ſolutionibus quibus anguli inveniuntur, trigonometricis multum poſt ponendae ſunt.

Coroll.

Coroll. III.

§. 344. Si anguli A et D, ponantur ſimul ſumti duobus rectis aequales prodeunte trapezio parallelarum baſium AB, DC aequatio pro diagonali eliminato ψ, in hanc mutatur:

$$f = \frac{\sqrt{(b^2 \text{ſin.} \lambda^2 \text{ſin.} \phi^2 + (d^2 \text{ſin.} \phi \mp b \text{ſin.} \lambda \cos. \phi)^2)}}{\text{ſin.} \phi}$$

ſed ϕ eliminato habetur:

$$f = \frac{(b^2 \text{ſin.} \lambda^2 \text{ſin.} \psi^2 + (d \text{ſin.} \psi \pm b \text{ſin.} \lambda \cos. \psi)^2)}{\text{ſin.} \psi}$$

Si vero anguli A et B ſimul ſumti ſint duobus rectis aequales, prodeunte trapezio parallelarum baſium AD, BC, habetur pro diagonali haec aequatio eliminato ψ

$$f = \frac{\sqrt{(b^2 \text{ſin.} \lambda^4 + (d \text{ſin.} \phi \mp b \text{ſin.} \lambda \cos. \lambda)^2)}}{\text{ſin.} \lambda}$$

ſed eliminato λ habetur:

$$f = \frac{\sqrt{(b \text{ſin.} \psi^4 + (d \text{ſin.} \phi \mp b \text{ſin.} \psi \cos. \psi)^2)}}{\text{ſin.} \psi}.$$

Si anguli diagonaliter oppoſiti B et D ſint ſimul ſumti aequales duobus rectis ut ſit trapezium circulo inſcriptibile, habetur pro diagonali haec aequatio ϕ eliminato

$$f = \frac{\sqrt{(b^2 \text{ſin.} \lambda^2 \text{ſin.} \psi^2 + (d \pm b \cos. \psi)^2 \text{ſin.} \lambda^2)}}{\text{ſin.} \psi}$$

vel etiam haec λ eliminato:

$$f = \frac{\sqrt{(b^2 \text{ſin.} \phi^2 \text{ſin.} \psi^2 + (d \pm b \cos \psi)^2 \text{ſin.} \psi^2)}}{\text{ſin.} \psi}.$$

Coroll. IV.

§. 345. In prima hypotheſi fit aequatio pro latere BC eliminato ψ

 $b=$

$$b = \frac{\sin.\phi\,(\surd(f^2 - d^2 \sin.\phi^2) \mp d\cos.\phi)}{\sin.\lambda}$$

sed eliminato ϕ habetur:

$$b = \frac{\sin.\psi\,(\surd(f^2 - d^2 \sin.\psi^2) \pm d\cos.\psi)}{\sin.\lambda}$$

sed in secunda hypothesi habetur eliminato ψ $b = \surd(f^2 - d^2 \sin.\phi^2) \pm d \sin.\phi \cot.\lambda$, sed eliminato λ fit: $b = \surd(f^2 - d^2 \sin.\phi^2) \pm d \sin.\phi \cot.\psi$, sed in prima hypothesi fit pro latere DC eliminato ψ, $d = \surd(f^2 - b^2 \sin.\lambda^2) \mp b \sin.\lambda \cot.\phi$, sed ϕ eliminato habetur: $d = \surd(f^2 - b^2 \sin.\lambda^2) \pm b \sin.\lambda \cot.\psi$. In altera hypothesi fit eliminato ψ

$$d = \frac{\sin.\lambda\,(\surd(f^2 - b^2 \sin.\lambda^2) \mp b\cos.\lambda)}{\sin.\phi},$$

si vero eliminetur λ habetur:

$$d = \frac{\sin.\psi\,(\surd(f^2 - b^2 \sin.\psi^2) \pm b\cos.\psi}{\sin.\phi},$$

in tertia hypothesi fit eliminato ϕ

latus $BC = b = \dfrac{\sin.\psi \surd(f^2 - d^2 \sin.\lambda^2) \pm d \sin.\lambda \cos.\psi}{\sin.\lambda}$

sed eliminato λ, $b = \dfrac{\sin.\psi \surd(f^2 - d^2 \sin.\phi^2) \pm d \sin.\phi \cos.\psi}{\sin.\phi}$

in eadem hypothesi fit pro latere DC eliminato ϕ,

$$d = \frac{\sin.\psi\,\surd(f^2 - b^2 \sin.\lambda^2 \pm b \sin.\lambda \cos.\psi}{\sin.\lambda}$$

sed λ eliminato habetur:

$$d = \frac{\sin.\psi \surd(f^2 - b^2 \sin.\phi^2 \pm b \sin.\phi \cos.\psi}{\sin.\psi}.$$

Coroll.

Coroll. V.

§. 346 In prima hypothſi fit pro angulo B eliminato ψ

$$fin.\lambda = \frac{fin.\,\varphi\,(\surd(f^2 - d^2\,fin.\,\varphi^2) \pm d\,cos.\,\varphi)}{b},$$

ſed φ eliminato fit

$$fin.\lambda = \frac{fin.\,\psi\,(\surd(f^2 - d^2\,fin.\,\psi^2) \pm\,cos.\,\psi)}{b},$$

conſequenter habetur:

$$\lambda = ang.\,fin.\frac{(fin.\,\varphi\,(\surd(f^2 - d^2\,fin.\,\varphi^2) \pm cos.\,\varphi))}{b}$$

vel etiam

$$\lambda = ang.\,fin.\frac{(fin.\,\psi\,(f^2 - d^2\,fin.\,\psi^2) \pm cos.\,\psi))}{b}.$$

In ſecunda hypotheſi fit pro angulo D, eliminato ψ

$$fin.\,\varphi = \frac{fin.\,\lambda\,(\surd(f^2 - b^2\,fin.\,\lambda^2) \mp b\,cos.\,\lambda)}{d}$$

ſed λ eliminato habetur:

$$fin.\,\varphi = \frac{fin.\,\psi\,(\surd(f - b\,fin.\,\psi^2) \pm b\,cos.\,\psi)}{d},$$

conſequenter ipſe angulus:

$$\varphi = ang.\,fin.\frac{(fin.\,\lambda\,(\surd(f^2 - b^2\,fin.\,\lambda^2) \mp b\,cos.\,\lambda))}{d}$$

vel etiam

$$\varphi = ang.\,fin.\frac{(fin.\,\psi\,(\surd(f^2 - b^2\,fin.\,\psi^2) \pm b\,cos.\,\psi))}{d}.$$

In tertia hypotheſi fit pro angulo A, φ eliminato,

$$tang.\,\psi = \frac{(b+d)\,(b-d)\,fin.\,\lambda}{d\surd(f^2 - d\,fin.\,\lambda^2) \pm b\surd(f^2 - b^2\,fin.\,\lambda^2)}$$

et ipſe angulus

$$\psi = ang.\ tang. \frac{(b+d)(b-d) fin. \lambda}{d\surd(f^2-d^2 fin.\lambda^2) \pm b\surd(f^2-b^2 fin.\lambda^2)}.$$

Coroll. VI.

§. 347. Si angulus A ponatur rectus vel tribus rectis aequalis, cadente vertice A ſupra diagonalem BD, intra vel extra triangulum BCD et trapezio inverſo prodeunte, aequatio pro diagonali in hanc mutatur: $f=\surd(b^2 fin.\lambda^2+d^2 fin.\phi^2)$, conſequenter in hoc caſu fit: $b=\frac{\surd(f^2-d^2 fin.\ \phi^2)}{fin.\ \lambda}$ et $d=\frac{\surd(f^2-b^2 fin.\ \lambda^2)}{fin.\ \phi}$. Si vero angulus D ponatur rectus vel tribus rectis aequalis, cadente D ad ſiniſtram diagonalis intra vel extra triangulum ABC et trapezio inverſo rurſus prodeunte, habetur pro diagonali haec aequatio:

$$f=\frac{\surd(b^2 fin.\ \lambda^2 + 2bd\, fin.\ \lambda\ cos.\ \psi + d^2)}{fin.\ \psi},$$

in hoc autem caſu fit pro latere BC

$b=\frac{fin.\psi\surd(f^2-d^2) \pm d\, cos.\psi}{fin.\ \lambda}$, ſed pro latere DC fit $d=fin.\psi\surd(f^2-b^2 fin.\lambda^2) \pm b\, fin.\lambda\, cos.\lambda$. Denique ſi angulus B ſit rectus vel tribus rectis aequalis cadente vertice B ad dextram diagonalis intra vel extra triangulum ADC, et trapezio, inverſo prodeunte; habetur pro diagonali haec aequatio: $f=\frac{\surd(b^2 fin.\psi^2+(d\, fin.\phi \pm b\, cos.\psi)^2)}{fin.\psi}$,

in

in hoc casu erit pro latere BC, $b = \sin. \psi \sqrt{(f^2 - d^2 \sin. \phi^2)} \pm d \sin. \phi \cos. \psi$ et pro latere DC, $d = \frac{\sin. \psi \sqrt{(f^2 - b^2)} \pm b \cos. \psi}{\sin. \phi}$.

Coroll. VII.

§. 348. Si ponatur angulus D, aequalis duobus rectis, trapezio in triangulum ABC abeunte prodit ex aequatione pro diagonali haec aequatio: $f \sin. \psi = b \sin. \lambda$, si vero ponatur angulus B aequalis duobus rectis, trapezio in triangulum ADC denuo abeunte, ex eadem aequatione habetur $f \sin. \psi = d \sin. \phi$, prior in hanc analogiam $f : \sin. \lambda = b : \sin. \psi$, sed posterior in hanc resolvitur, $f : \sin. \phi = d : \sin. \psi$; si vero ponatur angulus A aequalis duobus rectis, trapezio in triangulum BCD abeunte, ex utraque aequatione habetur $b \sin. \lambda = d \sin. \phi$, quae in hanc analogiam resolvitur, $b : \sin. \phi = d : \sin. \lambda$. His ergo consuetis analogiis trigonometricis, quia legitimae aequationes prodeunt, aequationes et operationes verificantur.

Coroll. VIII.

§. 349. Si tres anguli A, B, D, simul sumti ponantur duobus rectis aequales, et per consequens angulus quartus C etiam duobus rectis aequalis, ut trapezium abeat in triangulum. ABD; aequatio pro diagonali eliminato ψ in hanc abit:

$$f = \frac{\sqrt{(b^2 \sin. \lambda^2 \sin. (\lambda + \phi)^2 + (d \sin. \phi \mp b \sin. \lambda \cos. (\lambda + \phi))^2)}}{\sin. (\lambda + \phi}$$

 sed

fed eliminato λ habetur haec:

$$f=\frac{\sqrt{(b^2 \text{fin.}\psi^2 \text{fin.}(\phi+\psi)^2+(d \text{fin.}\phi \pm b \text{fin.}(\phi+\psi)\cos.\psi)^2)}}{\text{fin.}\psi}$$

et denique eliminato ϕ habetur haec:

$$f=\frac{\sqrt{(b^2 \text{fin.}\lambda^2 \text{fin.}\psi^2+(d \text{fin.}(\lambda+\psi) \pm b \text{fin.}\lambda \cos.\psi)^2)}}{\text{fin.}\psi}$$

Sed in hoc cafu fit eliminato ψ latus $BE=b=$

$$=\frac{\text{fin.}(\lambda+\phi)\sqrt{(f^2-d^2 \text{fin.}\phi^2)} \mp d \text{fin.}\phi \cos.(\lambda+\phi)}{\text{fin.}\lambda}$$

fed λ eliminato habetur haec:

$$b=\frac{\text{fin.}\psi\sqrt{(f^2-d^2 \text{fin.}\phi^2)} \pm d \text{fin.}\phi \cos.\psi}{\text{fin.}(\phi+\psi)},$$

et denique ϕ eliminato habetur:

$$b=\frac{\text{fin.}\psi\sqrt{(f^2-d^2 \text{fin.}(\lambda+\psi)^2)} \pm d \text{fin.}(\lambda+\psi)\cos.\psi}{\text{fin.}\lambda}$$

Similiter in hac hypothefi fit eliminato ψ, latus $DE=d=$

$$=\frac{\text{fin.}(\lambda+\phi)\sqrt{(f^2-b^2 \text{fin.}\lambda^2)} \mp b \text{fin.}\lambda \cos.(\lambda+\phi)}{\text{fin.}\phi}$$

fed λ eliminato habetur haec:

$$d=\frac{\text{fin.}\psi\sqrt{(f^2-b^2 \text{fin.}(\phi+\psi)^2} \pm b \text{fin.}(\phi+\psi)\cos.\psi)}{\text{fin.}\phi}$$

et denique eliminato ϕ habetur

$$d=\frac{\text{fin.}\psi\sqrt{(f^2-b^2 \text{fin.}\lambda^2)} \pm b \text{fin.}\lambda \cos.\psi}{\text{fin.}(\lambda+\psi)}.$$

Scholion III.

§. 350. Continent hae aequationes novas quasdam folutiones triangulorum, totalis ABD, in quo datur latus BD, et anguli duo vel eidem lateri

lateri adjacentes, vel etiam unus adjacens et alter oppositus, et partialium ABE, AED, in quibus dantur latera duo et unus angulus, hae autem solutiones vulgaribus trigonometricis parum vel nihil cedunt.

Coroll. IX.

§. 351. Si angulus C ponatur aequalis tribus rectis et consequenter summa angulorum A, B, D, uni recto aequalis, cadente C infra diagonalem BD, intra vel extra triangulum BAD et trapezio inverso prodeunte, habetur pro diagonali haec aequatio, ψ eliminato:

$$f = \frac{\sqrt{(b^2 \sin.\lambda^2 \cos.(\lambda+\phi)^2 + (d \sin.\phi \mp b \sin.\lambda \sin.(\lambda+\phi))^2)}}{\cos.(\lambda+\psi)}$$

λ vero eliminato habetur haec:

$$f = \frac{\sqrt{(b^2 \sin.\psi^2 \cos.(\phi+\psi)^2 + (d \sin.\phi \pm b \cos.(\phi+\psi) \cos.\psi)^2)}}{\sin.\psi}$$

et denique ϕ eliminato habetur haec:

$$f = \frac{\sqrt{(b^2 \sin.\lambda^2 \sin.\psi^2 + (d \cos.(\lambda+\psi) \pm b \sin.\lambda \cos.\psi)^2)}}{\sin.\psi}$$

sed eliminato ψ, habetur pro latere BC,

$$b = \frac{\cos.(\lambda+\psi) \sqrt{(f^2 - d^2 \sin.\phi^2)} \mp d \sin.\phi \sin.(\lambda+\phi)}{\sin.\lambda}$$

et latere $DC =$

$$d = \frac{\cos.(\lambda+\phi) \sqrt{(f^2 - b^2 \sin.\lambda^2)} \mp b \sin.\lambda \sin.(\lambda+\phi)}{\sin.\phi}$$

sed λ eliminato fit pro latere priore

$$b = \frac{\sin.\psi \sqrt{(f^2 - d^2 \sin.\phi^2)} \pm d \sin.\phi \cos.\psi}{\cos.(\phi+\psi)}$$

 et

et pro posteriore

$$d = \frac{fin.\psi \surd (f^2 - b^2 cos.(\phi+\psi)^2) \pm b\, cos.(\phi+\psi) cos.\psi}{fin.\phi}$$

et denique ϕ eliminato habetur pro priore,

$$b = \frac{fin.\psi \surd (f^2 - d^2 cos.(\lambda+\psi)^2) \pm d\, cos.(\lambda+\psi) cos.\psi}{fin.\lambda}$$

et pro posteriore,

$$d = \frac{fin.\psi \surd (f^2 - b^2 fin.\lambda^2) \pm b\, fin.\lambda\, cos.\psi.}{cos.(d+\psi)}$$

Coroll. X.

§. 352. Si in aequatione prima ponatur $f = d$, adhibito ſigno radicali affirmativo, aequatio mutatur in hanc: $b\, fin.\lambda = f\, fin.(\psi - \phi)$ vel etiam $b\, fin.\lambda = f\, fin.(\phi - \psi)$ verum adhibito ſigno radicali negativo fit $b\, fin.\lambda = -f\, fin.(\phi + \psi)$, ſimiliter ſi in aequatione altera ſit $f = b$, adhibito ſigno affirmativo prodit: $d\, fin.\phi = f\, fin.(\psi - \lambda)$ vel etiam $d\, fin.\phi = f\, fin.\phi = f\, fin.(\lambda - \psi)$; ſed negativo adhibito fit: $d\, fin.\phi = -f\, fin.(\lambda + \psi)$.

Scholion IV.

§. 353. Primae aequationes obtinent in hypotheſi figurae conſtructae, quoties unum triangulum ad dextram, alterum ad ſiniſtram diagonalis cadit; caeterae aequationes obtinent quoties utrumque triangulum cadit ad eandem partem diagonalis, hoc eſt quoties trapezium eſt inverſum, et ſecundae quidem locum habent ſi trapezium ſit totaliter inverſum, tertiae vero, ſi partialiter inverſum.

Schol.

Scholion V.

§. 354. Aequatio hujus problematis pro trapezio directo tam est simplex, ut nullas ex variatione figurae varietates recipere queat, nisi illas ipsas, quae ex ambiguitate signi radicalis et naturae cosinus oriuntur, quare si utrumque signum utrique termino dextri membri praefigatur, valet pro omni trapezio tam directo quam inverso, ut scribendum sit: $b \, \text{sin.} \, \lambda = \pm \, \text{sin.} \, \psi \sqrt{(f^2 - d^2 \, \text{sin.} \, \varphi^2)} \mp d \, \text{sin.} \, \varphi \, \text{cos.} \, \psi$.

Problema XXV.

§. 355. In figura quadrilatera proposita, $ABCD$, inter haec sex, latus $AB = a$, $CD = d$, diagonalem $AC = f$, et angulos $A = \psi$, $D = \varphi$, $B = \lambda$ aequationem invenire. Fig. XXV.

Solutio.

1) In praecedentis problematis solutione ex formatis sinu et cosinu anguli CAD, quorum ille $\frac{d \, \text{sin.} \, \varphi}{f}$; hic $\frac{\sqrt{(f^2 - d^2 \, \text{sin.} \, \varphi^2)}}{f}$ et ex sinu cosinuque anguli A inventus fuit,

$$\text{sin.} \, CAB = \frac{\text{sin.} \, \psi \sqrt{(f^2 - d^2 \, \text{sin.} \, \varphi^2)} - d \, \text{cos.} \, \psi \, \text{sin.} \, \varphi}{f}$$

ex quibus praeterea formatur:

$$\text{cos.} \, CAB = \frac{\text{cos.} \, \psi \sqrt{(f^2 - d^2 \, \text{sin.} \, \varphi^2)} + d \, \text{sin.} \, \varphi \, \text{sin.} \, \psi}{f}$$

2) Ex his et ex sinu cosinuque anguli B, formo sinum anguli tertii ACB, quem invenio esse:

sin.

$$\frac{\sin.\lambda\cos.\psi\sqrt{(f^2-d^2\sin.\phi^2)}+d\sin.\lambda\sin.\phi\sin.\psi}{f}$$
$$+\frac{\cos.\lambda\sin.\psi\sqrt{(f^2-d^2\sin.\phi^2)}-d\sin.\phi\cos.\lambda\cos.\psi}{f}$$
$$=\frac{\sin.(\lambda+\psi)\sqrt{(f^2-d^2\sin.\phi^2)}-d\sin.\phi\cos.(\lambda+\psi)}{f}=$$
$$=\sin.\ ACB.$$

3) Jam in triangulo finistro pervenio ad hanc analogiam: $AC:\sin.B=AB:\sin.ACB$, h. e.
$$f:\sin.\lambda=a:\frac{\sin.(\lambda+\psi)\sqrt{(f^2-d^2\sin\phi^2)}-d\sin.\phi\cos.(\lambda+\psi)}{f}$$
et consequenter habetur haec aequatio: $a\sin\lambda=\sin.(\lambda+\psi)\sqrt{(f^2-d^2\sin.\phi^2)}-d\sin.\phi\cos.(\lambda+\psi)$, Q. E. I.

Scholion 1.

§. 356. Altera solutio obtinetur incipiendo in triangulo finistro et ex finu anguli $ACB=\frac{a\sin.d}{f}$ et cofinu $\frac{\sqrt{(f^2-a^2\sin.\lambda^2)}}{f}$, nec non ex finu cofinuque anguli B formando
$$\sin.CAB=\frac{\sin.\lambda\sqrt{(f^2-a^2\sin.\lambda^2)}+a\sin.\lambda\cos.\lambda}{f}$$
et $$\cos.CAB=\frac{\cos.\lambda\sqrt{(f^2-a^2\sin.\lambda^2)}+a\sin.\lambda^2}{f}$$
et ex his atque finu cofinuque anguli A componendo $\sin.(\psi-CAB)=$
$$=\frac{-\sin.\psi\cos.\lambda\sqrt{(f^2-a^2\sin.\lambda^2)}+a\sin.\lambda^2\sin.\psi}{f}$$

$-\cos.$

$$\frac{-\cos.\psi\,\text{fin.}\,\lambda\,\sqrt{(f^2-a^2\,\text{fin.}\,\lambda^2)}-a\,\text{fin.}\,\lambda\cos.\lambda\cos.\psi}{f}$$

$$=\frac{-\text{fin.}(\lambda+\psi)\sqrt{(f^2-a^2\text{fin.}\lambda^2)}-a\,\text{fin.}\lambda\cos.(\lambda+\psi)}{f}=$$

$=\text{fin.}\,CAD$. His factis in triangulo dextro pervenio ad hanc analogiam: $AC: AC\,\text{fin.}\,D=CD:$ $\text{fin.}\,CAD$, h. e.

$$f:\text{fin.}\varphi=d:\frac{-\text{fin.}(\lambda+\psi)\sqrt{(f^2-a^2\text{fin}\lambda^2)}-a\text{fin.}\lambda\cos.(d+\psi)}{f}$$

et consequenter ad hanc aequationem: $d\,\text{fin.}\,\varphi$ $=-\text{fin.}(\lambda+\psi)\sqrt{(f^2-a^2\text{fin.}\lambda^2)}-a\,\text{fin.}\lambda\cos.(\lambda+\psi)$.

Scholion II.

§. 357. Si ex aequatione priore per extractionem radicis quadraticae quaeratur valor ipsius $d\,\text{fin.}\,\varphi$, prodit aequatio posterior adhibito signo radicali negativo $d\,\text{fin.}\,\varphi=-a\,\text{fin.}\,\lambda\cos.(\lambda+\psi)$ $\pm\text{fin.}(\lambda+\psi)\sqrt{(f^2-a^2\,\text{fin.}\,\lambda^2)}$ quare si ex posteriore per extractionem radicis quadraticae quaeratur $a\,\text{fin.}\,\lambda$, habetur aequatio prior, adhibito signo radicali affirmativo, igitur in aequatione posteriore terminum signo radicali affectum ex negativo in affirmativum mutare licet, aequationes autem formaliter tantum differunt.

Coroll. I.

§. 358. Ex priore aequatione statim liquet esse, latus $AB=a=$

$$=\frac{\text{fin}(\lambda+\psi)\sqrt{(f^2-d^2\text{fin.}\varphi^2)}\pm d\,\text{fin.}\,\varphi\cos.(\lambda+\psi)}{\text{fin.}\,\lambda}$$

quod idem ex posteriore radicem extrahendo obtinetur,

tinetur, ſed ex hac etiam ſtatim obtinetur: latus $DC = d =$

$$= \frac{\text{ſin.}(\lambda+\psi)\sqrt{(f^2 - a^2\,\text{ſin.}\,\lambda^2)} \pm a\,\text{ſin.}\,\lambda\cos.(\lambda+\psi)}{\text{ſin.}\,\phi}$$

Verum idem etiam ex priore radicem extrahendo obtinetur. At ex utraque aequatione idem pro diagonali valor hic elicitur:

$$f = \frac{\sqrt{(a^2\,\text{ſin.}\,\lambda^2 \pm 2ad\,\text{ſin.}\,\lambda\,\text{ſin.}\,\phi\cos.(\lambda+\psi) + d^2\,\text{ſin.}\,\phi^2)}}{\text{ſin.}(\lambda+\psi)}$$

vel etiam hic

$$f = \frac{\sqrt{(a^2\,\text{ſin}\lambda^2\,\text{ſin.}(\lambda+\psi)^2 + (d\,\text{ſin}\phi \pm a\,\text{ſin}\lambda\cos.(\lambda+\psi))^2)}}{\text{ſin.}(\lambda+\psi)}$$

Coroll. II.

§. 359. Ex poſteriore aequatione ſtatim incurrit in oculos eſſe pro angulo D obtinendo

$$\text{ſin.}\,\phi = \frac{\text{ſin.}(\lambda+\psi)\sqrt{(f^2 - a^2\,\text{ſin.}\,\lambda^2)} \mp a\,\text{ſin.}\,\lambda\cos(\lambda+\psi)}{d}$$

et ipſum angulum: $\phi = \text{ang. ſin.}$

$$\frac{(\text{ſin.}(\lambda+\psi)\sqrt{(f^2 - a^2\,\text{ſin.}\,\lambda^2)} \pm a\,\text{ſin.}\,\lambda\cos.(\lambda+\psi))}{d}$$

Sed pro angulo B obtinendo habetur haec aequatio:

$$\text{tang.}\,\lambda = \frac{\text{ſin.}\,\psi\sqrt{(f^2 - d^2\,\text{ſin.}\,\phi^2)} \pm d\,\text{ſin.}\,\phi\cos.\,\psi}{a - d\,\text{ſin.}\,\phi\,\text{ſin.}\,\psi - \cos.\,\psi\sqrt{(f^2 - d^2\,\text{ſin.}\,\phi^2)}}$$

vel etiam haec:

$$\text{tang.}\,\lambda = \frac{\text{tang.}\,\psi\sqrt{(f^2 - d^2\,\text{ſin.}\,\phi^2)} - d\,\text{ſin.}\,\phi}{a\,\text{ſec.}\,\psi - d\,\text{ſin.}\,\phi\,\text{tang.}\,\psi - \sqrt{(f^2 - d^2\,\text{ſin.}\,\phi^2)}}$$

et conſequenter ipſe angulus:

$$\lambda = \text{ang. tang.}\frac{\text{tang.}\,\psi\sqrt{(f^2 - d^2\,\text{ſin.}\,\phi^2)} - d\,\text{ſin.}\,\phi}{a\,\text{ſec.}\,\psi - d\,\text{ſin.}\,\phi\,\text{tang.}\,\psi - \sqrt{(f^2 - d^2\,\text{ſin.}\,\phi^2)}}$$

Si

Si aequationes ita combinentur ut prior per $d\,fin.\,\varphi$, posterior per $a\,fin.\,\lambda$ multiplicetur, pro obtinendo angulo A habetur haec aequatio:

$$tang.\,(\lambda+\psi) = \frac{(d\,fin.\,\varphi + a\,fin.\,\lambda)\,(d\,fin.\,\varphi - a\,fin.\,\lambda)}{d\,fin.\,\varphi\,\surd(f^2-d^2fin.\,\varphi^2) \pm a\,fin.\,\lambda\,\surd(f^2-a^2fin.\,\lambda^2)}$$

et ipse angulus, $\psi = ang.\,tang.$

$$\frac{(d\,fin.\,\varphi + a\,fin.\,\lambda)\,(d\,fin.\,\varphi - a\,fin.\,\lambda)}{d\,fin.\,\varphi\,\surd(f^2-\lambda^2 fin.\,\varphi^2) \pm a\,fin.\,\lambda\,\surd(f^2-a^2fin.\,\lambda^2)}.$$

Scholion III.

§. 360. Me non monente apparet, fieri posse, ut sit $a\,fin.\,\lambda$ majus, quam $d\,fin.\,\varphi$, et consequenter etiam scribi posse:

$$tang.\,(\lambda+\psi) = \frac{(d\,fin.\,\varphi + a\,fin.\,\lambda)\,(a\,fin.\,\lambda - d\,fin.\,\varphi)}{d\,fin.\,\varphi\,\surd(f^2-d^2fin.\,\varphi^2) \pm a\,fin.\,\lambda\,\surd(f^2-a^2fin.\,\lambda)}.$$

Caeterum inter sex illos casus qui in aequatione continentur, non datur nisi unus Tetragonometriae proprius, qui obtinet quando figura ex invenienda diagonali construenda proponitur, et in Geometria practica constituit problema novum utile simul et satis pulchrum. Caeteri quinque casus utrique methodo et trigonometricae et tetragonometricae pariter subsunt, quorum omnium solutiones evolvi, quia ex aequationibus haud difficulter eruuntur; solutiones pro lateribus datae trigonometricis parum cedunt, nec solutiones trigonometricae pro angulis tetragonometricas longo intervallo post se relinqunt.

Coroll.

Coroll. III.

§. 361. Si anguli A et B ponantur simul sumti aequales duobus rectis, ut prodeat trapezium parallelarum basium AD, BC, utraque aequatio in hanc mutatur: $a\, fin.\lambda = d\, fin.\phi$, unde haec analogia et hoc Theorema $a : fin.\phi = d : fin.\lambda$, uti supra. Si vero anguli A et D, ponantur simul duobus rectis aequales, ut prodeat trapezium parallelarum basium AB, DC, aequatio pro diagonali, angulo ϕ eliminato, haec habetur:

$$f = \frac{\sqrt{(a^2 fin.\lambda^2 fin.(\lambda+\psi)^2 + (d\, fin.\psi \pm a\, fin.\lambda\, cos.(\lambda+\psi))^2)}}{fin.(\lambda+\psi)}$$

sed eliminato angulo ψ habetur haec altera:

$$f = \frac{\sqrt{(a^2 fin.\lambda^2 fin.(\lambda-\phi)^2 + (d\, fin.\phi \pm a\, fin.\lambda\, cos.(\lambda-\phi))^2)}}{fin.(\lambda-\phi)}$$

In hac posteriore hypothesi habetur pro latere AB, ϕ eliminato,

$$a = \frac{fin.(\lambda+\psi)\sqrt{(f^2 - d^2 fin.\psi^2} \pm d\, fin.\psi\, cos.(\lambda+\psi)}{fin.\lambda}$$

At ψ eliminato habetur haec altera

$$a = \frac{fin.(\lambda-\phi)\sqrt{(f^2 - d^2 fin.\phi^2)} \pm d\, fin.\phi\, cos.(\lambda-\phi)}{fin.\lambda}$$

verum pro latere DC, habetur eliminato ϕ haec aequatio:

$$d = \frac{fin.(\lambda+\psi)\sqrt{(f^2 - a^2 fin.\lambda^2)} \pm a\, fin.\lambda\, cos.(\lambda+\psi)}{fin.\psi}$$

at ψ eliminato haec:

$$d = \frac{fin.(\lambda-\phi)\sqrt{(f^2 - a^2 fin.\lambda^2)} \pm a\, fin.\lambda\, cos.(\lambda-\phi)}{fin.\phi}$$

deni-

denique in hoc casu habetur pro angulo B, ϕ eliminato, $tang.\lambda = \frac{tang.\psi \surd(f^2 - d^2 fin.\psi^2) - d\, fin.\psi}{a\, sec.\psi - d\, fin.\psi\, tang.\psi - \surd(f^2 - d^2 fin.\psi^2)}$.

Coroll. IV.

§. 362. Si anguli diagonaliter oppositi B et D ponantur simul sumti duobus rectis aequales, ut prodeat trapezium circulo inscriptibile, habetur pro diagonali haec aequatio ϕ eliminato

$$f = \frac{fin.\lambda \surd(a^2 + d^2 \pm 2ad\, cos.(\lambda + \psi))}{fin.(\lambda + \psi)}$$

sed λ eliminato habetur:

$f = \frac{fin.\phi \surd(a^2 + d^2 \mp 2ad\, cos.(\psi - \phi))}{fin.(\psi - \phi)}$, sed in hoc casu habetur pro latere AB, ϕ eliminato haec aequatio: $a = \frac{fin.(\lambda + \psi)\surd(f^2 - d^2 fin.\lambda^2) \pm d\, fin.\lambda\, cos.(\lambda + \psi)}{fin.\lambda}$

λ vero eliminato habetur haec:

$$a = \frac{fin.(\psi - \phi)\surd(f^2 - d^2 fin\,\phi^2) \mp d\, fin.\phi\, cos.(\psi - \phi)}{fin.\phi},$$

sed pro latere DC. ϕ eliminato habetur haec aequatio: $d = \frac{fin.(\lambda + \psi)\surd(f^2 - a^2 fin.\lambda^2) \pm a\, fin.\lambda\, cos.(\lambda + \psi)}{fin.\lambda}$

at eliminato λ habetur:

$$d = \frac{fin.\cdot\psi - \phi)\surd(f^2 - a^2 fin.\phi^2) \mp a\, fin.\phi\, cos.(\psi - \phi)}{fin.\phi}.$$

In hoc autem casu fit pro angulo A

$$\psi = ang.\, tang. \frac{(a + d^2)(a - d)\, fin.\lambda}{d\surd(f^2 - d^2 fin.\lambda^2) \pm a\surd(f^2 - a^2 fin.\lambda^2)}$$

$$= ang. \frac{(a + d)(a - d)\, fin.\phi}{d\surd(f^2 - d^2 fin.\phi^2) \pm a\surd(f^2 - a^2 fin.\phi^2}.$$

Coroll. V.

§. 363. Sit angulus A rectus vel tribus rectis aequalis cadente vertice A supra diagonalem intra vel extra triangulum BCD, et trapezio inverso prodeunte, habetur pro diagonali haec aequatio:

$$f = \frac{\sqrt{(a^2 \sin.\lambda^2 \cos.\lambda^2 + (d \sin.\phi \mp a \sin.\lambda^2)^2)}}{\cos.\lambda}$$

sed in hoc casu habetur pro latere AB haec aequatio: $a = \cot.\lambda \sqrt{(f^2 - d^2 \sin.\phi^2} \mp d \sin.\phi$ sed pro latere DC habetur:

$$d = \frac{\cos.\lambda \sqrt{f^2 - a^2 \sin.\lambda^2}) \mp a \sin.\lambda^2)}{\sin.\phi}$$ sed in hoc casu fit,

$\tang.\lambda = \sqrt{(f^2 - d^2 \sin.\phi^2)} : (a \mp d \sin.\phi)$ et

$$\sin.\phi = \frac{\cos.\lambda \sqrt{(f^2 - a^2 \sin.\lambda^2)} \mp a \sin.\lambda^2}{d}.$$

Coroll. VI.

§. 364. Sit angulus D rectus vel tribus rectis aequalis, cadente vertice D ad sinistram diagonalis intra vel extra triangulum ABC, et trapezio inverso rursus prodeunte, habetur pro diagonali haec aequatio:

$$f = \frac{\sqrt{(a^2 \sin.\lambda^2 \sin.(\lambda+\psi)^2 + (d \mp a \sin.\lambda \cos.(\lambda+\psi))^2)}}{\sin.(\lambda+\psi)}$$

Sed in hoc casu habetur pro latere AB haec aequatio: $a = \dfrac{\sin.(\lambda+\psi)\sqrt{(f^2-d^2)} \mp d\cos.(\lambda+\psi)}{\sin.\lambda}$

sed pro latere DC habetur: $d = \sin(\lambda+\psi)$ $\sqrt{(f^2 - a^2 \sin.\lambda^2)} \pm a \sin.\lambda \cos.(\lambda+\psi)$ et in hoc casu fit, $\tang.\lambda = \dfrac{\tang.\psi \sqrt{(f^2-d^2)} \mp d}{a \sec.\psi \mp d \tang.\psi - \sqrt{(f^2-d^2)}}$

sed

ſed pro angulo A habetur:

$$\psi = ang.\,tang.\frac{(a\,fin.\,\lambda + d)\,(a\,fin.\,d - d)}{d\,\surd(f^2 - d^2) \pm a\,fin.\lambda\,\surd(f^2 - a^2\,fin.\lambda^2)} - \lambda.$$

Coroll. VII.

§. 365. Si angulus B ponatur rectus vel tribus rectis aequalis, cadente vertice B ad dextram diagonalis intra vel extra triangulum ADC, et trapezio inverſo prodeunte, habetur pro diagonali haec aequatio: $f = \frac{\surd(a^2\,cos.\,\psi^2 + (d\,fin.\,\phi \mp a\,fin.\,\psi)^2)}{cos.\,\psi}$

ſed in hoc caſu habetur pro latere $AB = a = cos.\,\psi\,\surd(f^2 - d^2 fin.\,\phi^2) \mp d\,fin.\,\phi\,fin.\,\psi$, et pro latere DC habetur: $d = \frac{cos.\,\psi\,\surd(f^2 - a^2) \mp a\,fin.\,\lambda\,fin.\,\psi}{fin.\,\phi}$,

In hoc caſu habetur pro angulo A haec aequatio:

$$\psi = ang.\,tang.\frac{(d\,fin.\,\phi + a)\,(d\,fin.\,\phi - a)}{d\,fin.\phi\,\surd(f^2 - d^2 fin.\,\phi^2 \pm a)\,\surd(f^2 - a^2)}.$$

Coroll. VIII.

§. 366. Sint anguli A et B ſimul ſumti uni recto vel tribus rectis aequales, aequatio pro diagonali in hanc mutatur: $f = \surd(a^2 fin.\lambda^2 + d^2 fin.\,\phi^2)$; in hoc itaque caſu fit, latus $AB = a = \frac{\surd(f^2 - d^2\,fin.\,\phi^2)}{fin.\,\lambda}$

et latus $DC = d = \frac{\surd(f^2 - a^2\,fin.\,d^2)}{fin.\,\phi}$. Si anguli A et D ſimul ſumti ſint uni vel tribus rectis aequales, habetur pro diagonali haec aequatio, ϕ eliminato:

$$f = \frac{\surd(a^2 fin.\lambda^2 \mp 2ad\,fin.\lambda\,cos.\psi\,cos(\lambda + \psi) + d^2 cos.\psi^2)}{fin.\,(\lambda + \psi)}$$

 ψ autem

ψ autem eliminato haec habetur:

$$f = \frac{\surd(a^2 fin. \lambda^2 \mp 2ad\, fin. \lambda\, fin. \phi\, fin. (\lambda - \phi) + d^2 fin. \phi^2)}{cos. (\lambda - \phi)}$$

Si anguli diagonaliter oppositi B et D, sint simul sumti uni recto vel tribus rectis aequales; habetur pro diagonali haec aequatio ϕ eliminato:

$$f = \frac{\surd(a^2 fin\, \lambda^2 \mp 2ad\, fin. \lambda\, cos. \lambda\, cos. (\lambda + \psi) + d^2 cos. \lambda^2)}{fin. (\lambda + \psi)}$$

sed λ eliminato:

$$f = \frac{\surd(a^2 cos. \phi^2 \mp 2ad\, fin. \phi\, cos. \phi\, fin. (\psi - \phi) + d\, fin. \phi^2)}{cos. (\psi - \phi)}$$

Coroll. IX.

§. 367. In prima hypothesi habetur pro latere AB, $a = \frac{\surd(f^2 - d^2\, fin. \phi^2}{fin. \lambda}$ et pro latere DC, $d = \frac{\surd(f^2 - a^2\, fin. \lambda^2)}{fin. \phi}$.

In secunda hypothesi, ϕ eliminato, erit latus $AB = a =$

$$= \frac{fin (\lambda + \psi) \surd(f^2 - d^2 cos. \psi^2) \mp d\, cos. \psi\, cos. (\lambda + \psi)}{fin. \lambda}$$

ψ autem eliminato habetur:

$$a = \frac{cos. (\lambda - \phi) \surd(f^2 - d^2 fin. \phi^2) \mp d\, fin. \phi\, fin. (\lambda - \phi)}{fin. \lambda};$$

sed latus $DC = d =$

$$= \frac{fin. (\lambda + \psi) \surd(f^2 - a^2 fin. \lambda^2) \pm a\, fin. \lambda\, cos. (\lambda + \psi)}{cos. \psi}$$

vel etiam

$$d = \frac{cos. (\lambda - \phi) \surd(f^2 - a^2\, fin. \lambda^2) \mp a\, fin. \lambda\, fin. (\lambda - \phi)}{fin. \phi}.$$

In

In tertia hypotheſi eliminato ϕ habetur: latus $AB = a =$

$$= \frac{\text{ſin.}(\lambda+\psi)\sqrt{(f^2-d^2\cos.\lambda^2)} \mp d\cos.\lambda\cos.(\lambda+\psi)}{\text{ſin.}\lambda}$$

ſed λ eliminato:

$$a = \frac{\cos.(\psi-\phi)\sqrt{(f^2-d^2\text{ſin.}\phi^2)} \mp d\,\text{ſin.}\phi\,\text{ſin.}(\psi-\phi)}{\cos.\phi}$$

et latus $DC =$

$$= \frac{\text{ſin.}(\lambda+\psi)\sqrt{(f^2-a^2\text{ſin.}\lambda^2)} \pm a\,\text{ſin.}\lambda\cos.(\lambda+\psi)}{\cos.\lambda}$$

vel etiam

$$d = \frac{\cos.(\psi-\phi)\sqrt{(f^2-a^2\cos.\phi^2)} \pm a\cos.\phi\,\text{ſin.}(\psi-\phi)}{\text{ſin.}\phi}.$$

Coroll. X.

§. 368. Si angulus A ponatur aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus, et trapezio in triangulum BCD abeunte; pro recta CE habetur haec aequatio:

$$f = \frac{\sqrt{(a^2\text{ſin.}\lambda^2 \mp 2ad\,\text{ſin}\,\lambda\cos.\lambda\,\text{ſin.}\phi + d^2\text{ſin.}\phi^2)}}{\text{ſin.}\lambda}.$$

In hoc autem caſu habetur pro ſegmento BE haec aequatio: $a = \sqrt{(f^2-d^2\text{ſin.}\phi^2)} \mp d\,\text{ſin.}\phi\cot.\lambda$, et pro latere $CD = d = \frac{\text{ſin.}\lambda(\sqrt{(f^2-a^2\text{ſin.}\lambda^2)} \mp a\cos.\lambda)}{\text{ſin.}\phi}$

In hac hypotheſi fit pro angulo B obtinendo $\text{tang.}\lambda = d\,\text{ſin.}\phi : (a \pm \sqrt{(f^2-d^2\text{ſin.}\phi^2)})$.

Scholion IV.

§. 369. His aequationibus nova quadam ratione obtinentur ſolutiones triangulorum totalis

talis BCD et partialium EBC, et CDE, quae ſolutionibus trigonometricis vulgaribus vix cedunt.

Coroll. XI.

§. 370. Si angulus B vel D ponatur aequalis duobus rectis, lateribus in diagonalem AC incidentibus et trapezio in triangulum ACD vel ABC abeunte, fit in caſu priore: $f\,\text{ſin.}(\lambda+\psi) = d\,\text{ſin.}\varphi$, at in poſteriore: $f\,\text{ſin.}(\lambda+\psi) = a\,\text{ſin.}\lambda$, inde reſultant hae conſuetae analogiae trigonometricae $f : \text{ſin.}\varphi = d : \text{ſin.}(\lambda+\psi)$, $f : \text{ſin.}\lambda = a : \text{ſin.}(\lambda+\psi)$, quo ipſo aequationes et operationes verificantur.

Coroll. XII.

§. 371. Si angulus C, ſit aequalis duobus rectis et conſequenter tres caeteri A, B, D, aequales duobus rectis, abeunte trapezio in triangulum ABD; aequatio pro diagonali in hanc abit: $\cos.(\lambda+\psi)$ et $\text{ſin.}(\lambda+\psi)$ eliminatis;

$$f = \frac{\sqrt{(a^2\,\text{ſin.}\lambda^2 \pm 2ad\,\text{ſin.}\lambda\,\text{ſin.}\varphi\cos.\varphi + d^2\,\text{ſin.}\varphi^2)}}{\text{ſin.}\varphi},$$

in hoc autem caſu fit:

$$\text{latus } AB = a = \frac{\text{ſin.}\varphi(f - d^2\,\text{ſin.}\varphi^2 \pm d\,\text{ſin.}\varphi\cos.\varphi)}{\text{ſin.}\lambda}$$

at pro ſegmentis DE habetur haec aequatio: $d = \sqrt{(f^2 - a^2\,\text{ſin.}\lambda^2)} \pm a\,\text{ſin.}\lambda\cot.\varphi$.

Scholion V.

§. 372. His aequationibus methodis minus uſitatis invenitur in triangulo ABD recta AE latus AB et ſegmentum ED lateris BD.

Coroll.

Coroll. XIII.

§. 373. Si angulus C ſit rectus vel etiam tribus rectis aequalis. ut ſumma angulorum A, B, D ſit recto vel tribus rectis aequalis, cadente in caſu priore vertice C infra diagonalem BD, et trapezio inverſo prodeunte; aequatio pro diagonali in hanc mutatur:

$$f = \frac{\sqrt{(a^2 \sin. \lambda^2 \mp 2ad \sin. \lambda \sin. \phi^2 + d \sin. \phi^2)}}{\cos. \phi},$$

$\sin. (\lambda + \psi)$ et $\cos. (\lambda + \psi)$ eliminatis; in hac hypotheſi habetur pro latere AB, iisdem eliminatis:

$$a = \frac{\cos. \phi \sqrt{(f^2 - d^2 \sin. \phi^2)} \mp d \sin. \phi^2}{\sin. \lambda},$$ ſed pro latere DC habetur: $d = \cot. \phi \sqrt{(f^2 - a^2 \sin. \lambda^2)} \mp a \sin. \lambda.$

Scholion VI.

§. 374. Quod ſi loco laterum AB, CD adhibeantur latera reliqua AD, BC, cum ſit

$$\sin. CAB = \frac{b \sin. \lambda}{f} \text{ et } \cos. CAB = \frac{\sqrt{f^2 - b^2 \sin. \lambda^2)}}{f}$$ et hinc

$$\sin. ACB = \frac{\sin. \lambda \sqrt{(f^2 - b^2 \sin. \lambda^2)} + b \sin. \lambda \cos. \lambda}{f}$$

et $$\cos. ACB = \frac{-\cos. \lambda \sqrt{(f^2 - b^2 \sin. \lambda^2)} + b \sin. \lambda^2}{f}$$

et praeterea $\sin. C = - \sin. (\lambda + \phi + \psi)$ et $\cos. C = \cos. (\lambda + \phi + \psi)$, ſequitur eſſe:

$$\sin. ACD = \sin. (C - ACB) = \frac{\cos. \lambda \sin. (\lambda + \phi + \psi)}{f}$$

$$\frac{\sqrt{(f^2 - b^2 \sin. \lambda^2)} - b \sin. \lambda^2 \sin. (\lambda + \phi + \psi)}{f}$$

 $-\sin.$

$$-\frac{\sin.\lambda\ \cos.(\lambda+\phi+\psi)\ \sqrt{(f^2-b^2\sin.\lambda^2)}-}{f}$$

$$\frac{-b\sin.\lambda\ \cos.\lambda\ \cos.(\lambda+\phi+\psi)}{f}=$$

$$=\frac{\sin.(\phi+\psi)\sqrt{(f^2-b^2\sin.\lambda^2)}-b\sin.\lambda\cos.(\phi+\psi)}{f},$$

hinc in dextro triangulo pervenio ad hanc analogiam: $AC:\sin.D=AD:\sin.ACD$, h. e. $f:\sin.\phi=c:$

$$\frac{\sin.(\phi+\psi)\sqrt{(f^2-b^2\sin.\lambda^2)}-b\sin.\lambda\cos.(\phi+\psi)}{f}$$

et consequenter ad hanc aequationem: $c\sin.\phi=\sin.(\phi+\psi)\sqrt{(f^2-b^2\sin.\lambda^2)}-b\sin.\lambda\cos.(\phi+\psi)$. Quod si vero incipiam in triangulo dextro et per angulum A in sinistrum transeam; ad hanc aequationem pervenio: $b\sin.\lambda=-\sin.(\phi+\psi)\sqrt{(f^2-c^2\sin.\phi^2)}-c\sin.\phi\cos.(\phi+\psi)$; haec autem ex priori prodit, si in illa $c\sin.\phi$ et $b\sin.\lambda$ loca sua permutent, quarum aequationum, cum eadem omnino forma sit, ac illarum priorum, vere assertum fuit a V. C. Lambert, hunc casum a priori non esse distinguendum, sed potius ambos in unum confundendos.

Coroll. XIV.

§. 375. Quatenus autem hic casus a priore distingui potest, pro lateribus et angulis habeantur hae expressiones:

$$b=\frac{\sin.(\phi+\psi)\sqrt{(f^2-c^2\sin.\phi^2)}-c\sin.\phi\cos.(\phi+\psi)}{\sin.\lambda}$$

$$c=\frac{\sin.(\phi+\psi)\sqrt{(f^2-b^2\sin.\lambda^2)}-b\sin.\lambda\cos.(\phi+\psi)}{\sin.\phi}$$

$f=$

$$f = \frac{\surd(b^2 \text{fin.}\lambda^2 + 2bc\,\text{fin.}\lambda\,\text{fin.}\varphi\,\cos.(\varphi+\psi) + c^2\,\text{fin.}\varphi^2)}{\text{fin.}\,(\varphi+\psi)}$$

$$\text{fin.}\lambda = \frac{\text{fin.}(\varphi+\psi)\surd(f^2 - c^2\,\text{fin.}\varphi^2) - c\,\text{fin.}\varphi\cos.(\varphi+\psi)}{b}$$

$$\text{tang.}\,\varphi = \frac{\text{tang.}\,\psi\surd(f^2 - b^2\,\text{fin.}\,\lambda^2) - b\,\text{fin.}\,\lambda}{c\,\text{fec.}\,\psi - b\,\text{fin.}\lambda\,\text{tang.}\psi - \surd(f^2 - b^2\text{fin.}\lambda^2)}$$

$$\text{tang.}\ \psi\,(\varphi+\psi) =$$

$$= \frac{(b\,\text{fin.}\lambda + c\,\text{fin.}\varphi)\,(b\,\text{fin.}\lambda - c\,\text{fin.}\varphi)}{b\,\text{fin.}\lambda\surd(f^2 - b^2\text{fin.}\lambda^2) - c\,\text{fin.}\varphi\surd(f^2 - c^2\text{fin.}\varphi^2)}$$

et ipſe angulus

$$\psi = \text{ang. tang.}$$

$$\frac{(b\,\text{fin.}\lambda + c\,\text{fin.}\varphi)\,(b\,\text{fin.}\lambda - c\,\text{fin.}\varphi) - \varphi}{b\,\text{fin.}\lambda\surd(f^2 - b^2\text{fin.}\lambda^2) - c\,\text{fin.}\varphi\surd(f^2 - c^2\,\text{fin.}\varphi^2)}$$

Scholion VII.

§. 376. Aequatio in ſolutione hujus problematis inventa pro trapezio directo, valet in hypotheſi figurae conſtructae, exiſtente angulo CAD acuto, ſed eodem obtuſo exiſtente et angulo B angulum A excedente valet haec aequatio:

$$a\,\text{fin.}\lambda = -\text{fin.}(\lambda-\psi)\surd(f^2 - d^2\text{fin.}\varphi^2 - d\,\text{fin.}\varphi\cos.(\lambda-\psi)$$

ſed angulo A angulum B excedente haec:

$$a\,\text{fin.}\lambda = \text{fin.}(\psi-\lambda)\surd(f^2 - d^2\text{fin}\varphi^2) - d\,\text{fin.}\varphi\cos.(\psi-\lambda)$$

Quod ſi trapezium ſit partialiter inverſum, ita ut triangulum ADC cadat intra triangulum ABC, reproducitur aequatio in reſolutione inventa, ſi vero ſit totaliter inverſum cadente latere AD extra latus AB, ſed latere CD intra latus CB, et exiſtente angulo B majore angulo A, prodit aequatio allatarum prior cum

signis contrariis dextri membri. Sed existente angulo A majore angulo B, reproducitur allatarum posterior cum signis dextri membri contrariis, neque praeterea in hac aequatione varietates occurrunt, quomodo varietur figura, exceptis iis, quae ex ambigua natura signi radicalis & cosinus oriuntur; est igitur aequatio generalissima talis: $a\, fin.\, \lambda = \pm fin. \binom{\lambda \pm \psi}{\psi - \lambda}$

$$\sqrt{(f^2 - d^2 fin.\, \varphi^2)} \mp d\, fin.\, \varphi\, cos. \binom{\lambda \pm \psi}{\psi - \lambda}.$$



CAPUT

CAPUT VII.

Continens tria problemata primae classis particularis, sub posteriore principali contentae.

Problema XXVI.

§. 377. In proposita figura quadrilatera $ABCD$, inter haec sex: latus $AD=c$, $BC=b$, et angulus $A=\psi$, $D=\varphi$, $ACB=\alpha$ et $ACD=\beta$, aequationem invenire: Fig. XXVI.

Solutio.

1) In triangulo dextro formo hanc analogiam: $\text{sin}.\ ACD\ (\text{sin}.\ \beta) : AD\ (c) = \text{sin}.\ D\ (\text{sin}.\ \varphi) : AC = \frac{c\ \text{sin}.\ \varphi}{\text{sin}.\ \beta}$.

2) Cum data sint nomina primitiva angulorum D, et ACD, erit nomen derivativum anguli tertii $CAD = \pi - \beta - \varphi$, siquidem uti solet nomen primitivum duorum rectorum sit π, et ob datum nomen primitivum anguli A, quod est ψ, habetur nomen derivativum anguli CAB, $\psi + \beta + \varphi - \pi$, et hinc erit $\text{sin}.\ CAB = -\text{sin}.\ (\beta + \varphi + \psi)$ denique erit nomen derivativum anguli B, $2\pi - \beta - \alpha - \varphi - \psi$ et hinc $\text{sin}.\ B = -\text{sin}.\ (\alpha + \beta + \varphi + \psi)$.

3) His jam factis in triangulo sinistro pervenio ad hanc analogiam, $AC : \text{sin}.\ \beta = BC : \text{sin}.\ CAB$ h. e.

h. e. $\frac{c\, fin.\, \phi}{fin.\, \beta} : -fin.\, (\alpha + \beta + \phi + \psi) = b : -fin.\, (\beta + \phi + \psi)$, et confequenter habetur haec aequatio : $b\, fin.\, \beta\, fin.\, (\alpha + \beta + \phi + \psi) = c\, fin.\, \phi\, fin.\, (\beta + \phi + \psi)$, Q. E. I.

Solutio altera.

1) Ab angulis D et C in latus oppofitum AB demittantur normales Cc, Da, et ab angulo D ad perpendiculum Cc agatur parallela lateri AB, quae igitur utrique perpendiculo erit normalis, hinc angulus apud m, rectus, rectae mc, Da aequales inter fe, et confequenter triangulum DmC rectangulum.

2) His factis cum fit (per n. 2. fol. antec.) $fin.\, B = -fin.\, (\alpha + \beta + \phi + \psi)$, erit perpendiculum $Cc = -b\, fin.\, (\alpha + \beta + \phi + \psi)$, praeterea perpendiculum $Da = c\, fin.\, \psi = mc$, et hinc habetur: $Cm = -b\, fin.\, (\alpha + \beta + \phi + \psi) - c\, fin.\, \psi$.

3) In triangulo dextro formo hanc analogiam: $fin.\, ACD\, (fin\, \beta) : AD\, (c) = fin.\, CAD\, (fin.\, (\beta + \phi)) : CD = \frac{c\, fin.\, (\beta + \phi)}{fin.\, \beta}$, fed ob parallelas AB, Dm, erit angulus $CDm = \psi + \phi - \pi$, et hinc $fin.\, CDm = -fin.\, (\phi + \psi)$.

4) Nunc in triangulo rectangulo cDm, pervenio ad hanc analogiam: $fin.\, m\, Dc : mC = fin.\, m : Dc$, h. e. fubftitutis fymbolis: $-fin.\, \phi + \psi : -b\, fin.\, (\alpha + \beta + \phi + \psi) - c\, fin.\, \psi = 1 : \frac{c\, fin.\, (\beta + \phi)}{fin.\, \beta}$

et

et conſequenter ad hanc aequationem pervenio:
$b \sin.\beta \sin.(\alpha+\beta+\varphi+\psi)+c \sin.\psi \sin.\beta=c \sin.(\beta+\varphi) \sin.(\varphi+\psi)$, Q. E. I.

Coroll. I.

§. 378. Ex aequatione priore habetur: latus $BC = b = \frac{c \sin.\varphi \sin(\beta+\varphi+\psi)}{\sin.\beta \sin.(\alpha+\beta+\varphi+\psi)}$ et ex poſteriore habetur: $b=\frac{c \sin.(\beta+\varphi) \sin.(\varphi+\psi)-c \sin.\beta \sin.\psi}{\sin.\beta \sin.(\alpha+\beta+\varphi+\psi)}$
Sed ex priore eſt latus $AD = c$ $= b \sin.\beta \sin.(\alpha+\beta+\varphi+\psi) : \sin.\varphi \sin.(\beta+\varphi+\psi)$
et ex poſteriore $c=\frac{b \sin.\beta \sin.(\alpha+\beta+\varphi+\psi)}{\sin.(\beta+\varphi) \sin.(\varphi+\psi)-\sin.\beta \sin.\psi}$

Coroll. II.

§. 379. Ex priore aequatione habetur pro obtinendo angulo ACB,
$$\alpha = ang.\sin.\frac{c \sin.\varphi \sin.(\beta+\varphi+\psi)}{b \sin.\beta}-\beta-\varphi-\psi$$
et eſt poſteriore
$$\alpha = ang.\sin.\frac{c \sin(\beta+\varphi) \sin.(\varphi+\psi)-c \sin.\beta \sin.\psi}{b \sin.\beta}-\beta-\varphi-\psi$$
Pro angulo A obtinendo habetur ex aequatione priore
$$tang.\psi=\frac{c \sin.\varphi \sin.(\beta+\varphi)-b \sin.\beta \sin.(\alpha+\beta+\varphi)}{b \sin.\beta \cos.(\alpha+\beta+\varphi)-c \sin.\varphi \cos.(\beta+\varphi)}$$
ſed ex poſteriore habetur:
$$tang.\psi = \frac{c \sin.\varphi \sin.(\beta+\varphi)-b \sin.\beta \sin.(\alpha+\beta+\varphi)}{b \sin.\beta \cos.(\alpha+\beta+\varphi)+c \sin.\beta-c \sin.(\beta+\varphi) \cos.\varphi}$$

Coroll.

Coroll. III.

§. 380. Posito ad abbreviandum *sin.* $\beta = h$, *sin.*$(\beta+\psi) = m$, et *cos.*$(\beta+\psi) = n$, *sin.*$(\alpha+\beta+\psi) = p$, et *cos.*$(\alpha+\beta+\psi) = q$, *sin.* $\phi = x$ et *cos.* $\phi = \surd(1-x^2)$, facta sinuum separatione substitutione et transpositione, habetur haec aequatio abbreviata: $bhqx - cnx^2 = (cmx - bhp)\surd(1-x^2)$ ex qua sumtis quadratis, facta concinnatione et transpositione, solutio pro inveniendo angulo D est ad hanc aequationem quarti gradus perducta: $x^4 - \frac{2bh(nq+mp)x^3}{c} + \frac{(b^2h^2 - c^2m^2)}{c^2} x^2 + \frac{2bhmpx}{c} - \frac{b^2h^2p^2}{c^2} = o$, Sed quoniam est $nq + mp = cos.\,\alpha = k$ brevitatis ergo erit aequatio paulo contractior hujusmodi: $x^4 - \frac{2bhkx^3}{c} + \frac{(b^2h^2 - m^2)x^2}{c^2} + \frac{2bhmpx}{c} - \frac{b^2h^2p^2}{c^2} = o$. Posito iterum ad abbreviandum *sin.* $\phi = h$, *sin.*$(\phi+\psi) = m$, *cos.*$(\phi+\psi) = n$, *sin.*$(\alpha+\phi+\psi) = p$, *cos.*$(\alpha+\phi+\psi) = q$, *sin.* $\beta = x$, et *cos.* $\beta = \surd(1-x^2)$, facta separatione, substitutione et transpositione habetur haec aequatio abbreviata: $bqx^2 - chnx = (chm - bpx)\surd(1-x^2)$ ex qua sumtis quadratis, facta concinnatione et transpositione, solutio pro inveniendo angulo ACD est ad hanc aequationem quarti gradus perducta, cum sit uti prius, $nq + mp = cos.\,\alpha = k$, $x^4 - \frac{2chkx^3}{b} + \frac{(c^2h^2 - p^2)x^2}{b^2} + \frac{2chmpx}{b} - \frac{c^2h^2m^2}{b^2} = o$.

Schol.

Scholion I.

§. 381. Haec aequatio poſterior ex priore prodit, ſi c et b, φ et β pro ſe invicem ſubſtituantur. Inter ſex autem caſus qui in aequatione continentur duo dantur Tetragonometriae proprii, qui obtinent quando figura ex inveniendo angulo D vel ACD conſtruenda proponitur, quorum caſuum ſolutiones deductae ſunt ad aequationes quarti gradus, ſuppeditant vero problemata in Geometria practica nova utilia et ſatis pulchra. Caeteri caſus quatuor, utrique methodo et tetragonometricae et trigonometricae pariter ſubſunt. Solutiones pro lateribus datae, trigonometricis nihil, pro angulis A et ACB, non adeo multum cedunt.

Coroll. IV.

§. 382. Si anguli D, ACB, et ACD, hoc eſt anguli D et C, ponantur ſimul ſumti aequales duobus rectis ut prodeat trapezium parallelarum baſium AD, BC, aequatio prior pro latere BC in hanc mutatur: $b = \frac{c\, fin.\, \varphi\ fin.\, (\psi - \alpha)}{fin.\, \beta\ fin.\, \psi}$, prior autem pro latere AD in hanc: $c = \frac{b\, fin.\, \beta\, fin.\, \psi}{fin.\, \varphi\, fin.\, (\psi - \alpha)}$, prior pro angulo A in hanc abit: $tang.\ \psi = \frac{c\, fin.\, \alpha\ fin.\, \varphi}{c\, fin.\, \varphi\, cos.\, \alpha - b\, fin.\, \beta}$.

Coroll. V.

§. 383. Si anguli diagonaliter oppoſiti A et C ponantur aequales duobus rectis ut prodeat quadri-

drilaterum circulo inſcriptibile, aequatio prior pro latere BC in hanc mutatur: $b = \frac{c\, fin.(\phi - \alpha)}{fin.\, \beta}$,
In hoc autem caſu aequatio pro angulo D, in hanc mutatur: $\frac{x^2 + 2bhkx +}{c} \frac{b^2h^2 - c^2m^2}{c^2} = o$ ex qua extracta radice habetur:

$$x = \frac{\mp\, bhk \pm \surd(c^2 m^2 - b^2 h^2\, fin.\, \alpha^2)}{c}$$

ſive valoribus reſtitutis:

$$fin.\phi = \frac{\mp b\, fin.\beta \cos.\alpha \pm \surd(c^2 fin(\beta+\psi)^2 - b^2 fin.\alpha^2 fin.\beta^2)}{c}$$

Cum autem ſit per poſitionem trium angulorum $\alpha+\beta+\psi=\pi$, $fin.(\beta+\psi) = fin.\,\alpha$; aequatio in hanc mutatur: $fin.\phi = \frac{\mp b\, fin.\beta \cos.\alpha \pm fin.\alpha \surd(c^2 - b^2 fin.\beta^2)}{c}$.

Coroll. VI.

§. 384. Si angulus A ſit aequalis duobus rectis, lateribus AD, AB, in diagonalem BD incidentibus, et trapezio in triangulum BCD abeunte, habetur pro latere BC ex aequatione priori haec expreſſio: $b = \frac{c\, fin.\, \phi\; fin.\,(\beta+\phi)}{fin.\, \beta\; fin.\,(\alpha+\beta+\phi)}$
conſequenter pro ED haec: $c = \frac{b\, fin.\beta\, fin.(\alpha+\beta+\phi)}{fin.\,\phi\; fin.\,(\beta+\phi)}$,
ſed pro angulo ECB habetur haec:
$\alpha = ang.\, fin. \frac{c\, fin.\,\phi\; fin.\,(\beta+\phi)}{b\; fin.\,\beta} - \beta - \phi$. Caeterum in aequatione pro angulo D fit: $fin.\,(\beta+\psi) = -fin.\beta = -m = -h$, $\cos.(\beta+\psi) = -\cos.\beta = -n$, $fin.(\alpha+\beta+\psi)$ $=-$

$= - \text{fin.}\ (\alpha + \beta) = - p$, et $\cos.\ (\alpha + \beta + \psi) = - \cos.\ (\alpha + \beta) = - q$. Sed in aequatione pro angulo ACD fit $\text{fin.}\ (\phi + \psi) = - \text{fin.}\ \phi = - m = - h$, $\cos.\ (\phi + \psi) = - \cos.\ \phi = - n$, $\text{fin.}\ (\alpha + \phi + \psi) = - \text{fin.}\ (\alpha + \phi) = - p$, $\cos.\ (\alpha + \phi + \psi) = - \cos.\ (\alpha + \phi) = - q$, aequationes igitur, eliminato h, tales prodeunt prior: $x^4 + \frac{2bmkx^3}{c}$

$$+ \frac{(b^2 - c^2) m^2 x^2}{b^2} + \frac{2bm^2px}{c} - \frac{b^2m^2p^2}{c^2} = o$$

posterior: $x^4 \pm \frac{2cmkx^3}{b} + \frac{(c^2m^2 - b^2p^2)\,x^2}{b^2}$

$$+ \frac{2cm^2p + c^2m^4}{b^2} = o.$$

Coroll. VII.

§. 385. Si angulus C fit aequalis duobus rectis, lateribus CB, CD in diagonalem BD incidentibus, et trapezio in triangulum ABD abeunte; aequatio pro latere BE prior in hanc mutatur: $b = \frac{c\,\text{fin.}\ \phi\ \text{fin.}\ (\phi + \psi + \beta)}{\text{fin.}\ \beta\ \text{fin.}\ (\phi + \psi)}$, et consequenter latus $AD = c = \frac{b\,\text{fin.}\ \beta\ \text{fin.}\ (\phi + \psi)}{\text{fin.}\ \phi\ \text{fin.}\ (\beta + \phi + \psi)}$, et pro angulo A habetur haec expressio:

$$\text{tang.}\ \psi = \frac{- c\ \text{tang.}\ \phi\ (\text{fin.}\ (\beta + \phi) + b\ \text{fin.}\ \beta)}{b\ \text{fin.}\ \beta + \text{tang.}\ \phi\ \cos.\ (\beta + \phi)}.$$

Verum aequatio pro angulo D invariata manet, nisi quod fit $p = - \text{fin.}\ \psi$ et consequenter quartus terminus fiat negativus.

Scholion II.

§. 386. Corollarium penultimum continet pro lateribus ſolutiones trigonometricas analytice expreſſas, aequationes autem pro angulis datae, novas ſolutiones continent, anguli autem D, et ACD trigonometrice prorſus inveniri nequeunt, cum in neutro partialium habeantur, niſi duo data, nec in totali dentur niſi latus et angulus in uno caſu oppoſitus in altero adjacens, et pars ED lateris BD. Ultimum corollarium habet etiam ſolutiones pro lateribus trigonometricas analytice expreſſas, angulus autem ACD nova quadam ratione invenitur etiamſi trigonometrice quoque inveniri poſſit, angulus vero D nova ratione prorſus per aequationem quarti gradus invenitur, nec trigonometrice obtineri poteſt, cum in neutro triangulo partiali dentur in hoc caſu niſi duo, nec in ipſo totali niſi duo, cum parte BE lateris BD.

Coroll. VIII.

§. 387. Si angulus A ſit rectus vel tribus rectis aequalis, cadente vertice A ſupra diagonalem intra vel extra triangulum BCD et trapezio inverſo prodeunte ſit:

latus $BC = b = \frac{c\ \text{ſin.}\ \varphi\ \cos.\ (\beta+\varphi)}{\text{ſin.}\ \beta\ \cos.\ (\alpha+\beta+\varphi)}$ et

latus $DC = c = \frac{b\ \text{ſin.}\ \beta\ \cos.\ (\alpha+\beta+\varphi)}{\text{ſin.}\ \varphi\ \cos.\ (\beta+\varphi)}$ ſed angulus

ACB ſit $\alpha = ang.\ \cos.\frac{c\ \text{ſin.}\ \varphi\ \cos.\ (\beta+\varphi)}{b\ \text{ſin.}\ \beta} - \beta - \varphi$.

Aequatio quarti gradus coroll. III. quoad formam externam invariata manet, ſed in ipſa eſt

$\pm \cos.$

$\pm cos.\beta = \pm m$, $\pm cos.(\alpha+\beta) = \pm p$, similiter in aequatione quarti gradus pro inveniendo angulo ACD fit: $\pm cos.\varphi = \pm m$, et $\pm cos.(\alpha+\varphi) = \pm p$, quae etiam quo ad formam externam invariata manet.

Coroll. IX.

§. 388. Si angulus D sit rectus vel etiam aequalis tribus rectis, cadente vertice D ad sinistram diagonalis intra vel extra triangulum ABC, et trapezio inverso denuo prodeunte erit:

latus $BC = b = \frac{\pm c\ cos.(\psi+\beta)}{sin.\beta\ cos.(\alpha+\beta+\psi)}$ et consequenter latus $AD = c = \frac{\pm b\ sin.\beta\ cos.(\alpha+\beta+\psi)}{cos.(\psi+\beta)}$, sed angulus $ACB = \alpha = ang.\ cos.\frac{\pm c\ cos.(\psi+\beta)}{b\ sin.\beta} - \beta - \psi$ et pro angulo ψ habetur:

$tang.\psi = \frac{c \pm b\ tang.\beta\ cos.(\alpha+\beta)}{\pm tang.\beta\ (b\ sin.(\alpha+\beta) - c)}$. In aequatione vero quarti gradus pro angulo ACD fit: $sin.\varphi = \pm 1 = h$, et $\pm cos.\psi = m$, et $\pm cos.(\alpha+\psi) = p$, et aequatio paululum immutata fit hujusmodi:

$$x^4 \mp \frac{2ckx^3}{b} + \frac{(c^2-bp^2)x^2}{b^2} + \frac{2cmpx}{b} - \frac{c^2m^2}{b^2} = 0.$$

Coroll. X.

§. 389. Si angulus C sit rectus vel etiam tribus rectis aequalis, cadente vertice C infra diagonalem BD, intra vel extra triangulum BCD, et trapezio inverso prodeunte; pro latere BC habetur

betur haec expressio: $b = \pm \frac{c\,\text{sin.}\,\phi\,\text{sin.}\,(\beta+\phi+\psi)}{\text{sin.}\,\beta\,\text{cos.}\,(\phi+\psi)}$,
et latus $AD = c = \frac{\pm\, b\,\text{sin.}\,\beta\,\text{cos.}\,(\phi+\psi)}{\text{sin.}\,\phi\,\text{sin.}\,(\beta+\phi+\psi)}$,
sed pro angulo A habetur:
$tang.\,\psi = \frac{c\,\text{sin.}\,\phi\,\text{sin.}\,(\beta+\phi) \pm b\,\text{sin.}\,\beta\,\text{cos.}\,\phi}{\text{sin.}\,\phi\,(\mp b\,\text{sin.}\,\beta - c\,\text{cos.}\,(\beta+\phi))}$, vel
etiam $tang.\,\psi = \frac{c\,tang.\,\phi\,\text{sin.}\,(\beta+\phi) \pm b\,\text{sin.}\,\beta}{tang.\,\phi\,(\mp b\,\text{sin.}\,\beta - c\,\text{cos.}\,(\beta+\phi))}$,
in aequatione pro angulo D fit: $\text{cos.}\,\psi = \pm p$, et consequenter terminus quartus signo utroque erit efficiendus, quoad caetera manet aequatio invariata.

Coroll. XI.

§. 390. Denique si $-\text{sin.}\,(\alpha+\beta+\phi+\psi)$, ponatur aequalis recto vel tribus rectis, cadente vertice B, ad dextram diagonalis extra vel intra triangulum ACD, et trapezio inverso prodeunte, erit: latus $BC = b = \frac{c\,\text{sin.}\,\phi\,\text{sin.}\,(\beta+\phi+\psi)}{\text{sin.}\,\beta} = \frac{c\,\text{sin.}\,\phi\,\text{cos.}\,\alpha}{\text{sin.}\,\beta}$
et $c = b\,\text{sin.}\,\beta : \text{sin.}\,\phi\,\text{cos.}\,\alpha$, angulorum autem ternis datis, quartus etiam datur.

Scholion III.

§. 391. Aequationes supra datae valent pro trapezio in hypothesi figurae constructae, si nimirum unum triangulum a dextra, alterum a sinistra diagonalis cadat. Quod si vero trapezium fiat inversum ut triangulum utrumque ad eandem partem diagonalis cadat, ex. gr. ad sinistram, sitque partialiter tantum inversum, ut

totum

totum ADC, cadat intra triangulum ABC; pervenitur ad hanc aequationem: $b\, fin.\, \beta\, fin. (\beta+\varphi-\alpha-\psi) = c\, fin.\, \varphi\, fin.\, (\psi-\beta-\varphi)$. Sin autem fit totaliter inverfum, ut latus AD cadat extra latus AB; ad hanc aequationem pervenitur: $b\, fin.\, \beta\, fin. (\beta+\varphi+\psi-\alpha) = c\, fin.\, \varphi\, fin. (\psi+\beta+\varphi)$; eadem cum hac pofteriore prodit aequatio, fi trapezium ita fit partialiter inverfum, ut totum triangulum ADC cadat extra triangulum ABC, et hoc ambitu fuo comprehendat; verum eadem cum priori aequatio prodit, fi trapezium ita fit totaliter inverfum, ut latus CD cadat extra latus CB. Si utrumque triangulum cadat ad dextram diagonalis, triangulum vero ADC totum intra triangulum ABC, ut trapezium fit partialiter inverfum, prodit aequatio prorfus eadem cum prima. Si vero triangulum ABC totum fit intra triangulum ADC, prodit aequatio prorfus eadem cum fecunda. Si trapezium fit totaliter inverfum ita ut latus CD cadat extra latus CB, denuo prodit aequatio prima, fin autem fit totaliter inverfum ita ut latus CB cadat extra latus CD, prodit aequatio altera. Aequatio igitur pro trapezio inverfo generalis haec eft: $b\, fin.\, \beta\, fin. (\beta+\varphi\pm\psi-\alpha) = c\, fin.\, \varphi\, fin. (\psi\pm\beta\pm\varphi)$ et confequenter in hoc cafu pro trapezio generaliffima haec: $b\, fin.\, \beta\, fin.\, (\beta+\varphi\pm\psi\pm\alpha) = c\, fin.\, \varphi\, fin. (\psi\pm\beta\pm\varphi)$.

Coroll. XII.

§. 392. Ex hac aequatione valores laterum generales ftatim in oculos incurrunt, eft autem angulus

$$ACB=\alpha=ang.\,fin.\frac{c\,fin.\,\phi\,fin.(\psi\pm\beta\pm\phi)}{b\,fin.\,\beta}-\beta-\phi\mp\psi,$$

pro angulo autem A habetur:

$$tang.\,\psi=\frac{c\,fin.\,\phi\,fin.(\beta+\phi)-b\,fin.\,\beta\,fin.(\beta+\phi\pm\alpha)}{b\,fin.(\beta+\phi\pm\alpha)-c\,fin.\,\phi\,cos.(\beta+\phi)}$$

In aequatione pro angulo D, fit $m = fin.(\psi\pm\beta)$, $cos.(\psi\pm\beta)=n$, $fin.(\phi\pm\psi\pm\alpha) = p$, et $cos.(\beta\pm\psi\pm\alpha)=q$, fed in aequatione altera eft: $fin.(\psi\pm\phi)=m$, $cos.(\psi\pm\phi)=n$, $fin.(\phi\pm\psi\pm\alpha)=p$, et $cos.(\phi\pm\psi\pm\alpha)=q$, quoad caetera aequationum quardi gradus forma manet invariata, quia $mp-nq$ fit ut prius $cos.\,\alpha$.

Problema XXVII.

Fig. XXVII. §. 393. In figura quadrilatera propofita $ABCD$ inter haec fex: latus $BC=b$, $CD=d$, $AD=c$ et angulos $A=\psi$, $ACB=a$, $ACD=\beta$; aequationem invenire.

Solutio.

1) Ducatur diagonalis AC, et ab angulis B, D, demittantur in ipfam perpendicula Bb, Dd, et erit perpendiculum $Bb=b\,fin.\,\alpha$ et $Cb=b\,cos.\,\alpha$; fed perpendiculum $Dd=d\,fin.\,\beta$, recta $Cd=d\,cos.\,\beta$, et confequenter $db=d\,cos.\,\beta-b\,cos.\,\alpha$, fed eft etiam $Ad=\surd(c^2-d^2\,fin.\,\beta^2)$ et hinc obtinetur: $Ab=d\,cos.\,\beta-b\,cos.\,\alpha+\surd(c^2-d^2\,fin.\,\beta^2)$.

2) Cum fit per antecedentia:

$$fin.\,CAB=\frac{fin.\,\psi\surd(c^2-d^2\,fin.\,\beta^2)-d\,fin.\,\beta\,cos.\,\psi}{c}$$

et

$$cos.\,CAB=\frac{cos.\,\psi\surd(c^2-d^2\,fin.\,\beta^2)+d\,fin.\,\beta\,fin.\,\psi}{c}$$

In

In triangulo rectangulo ABb, formo hanc analogiam $Bb : \text{fin.}\, CAB = AB : \cos.\, CAB$, h. e. $b\,\text{fin.}\,\alpha : \text{fin.}\,\psi \sqrt{(c^2 - d^2\,\text{fin.}\,\beta^2)} - d\,\text{fin.}\,\beta\,\cos.\,\psi = d\cos.\,\beta - b\cos.\,\alpha + \sqrt{(c^2 - d^2\,\text{fin.}\,\beta^2)} : \cos.\,\psi \sqrt{(c^2 - d^2\,\text{fin.}\,\beta^2)} + d\,\text{fin.}\,\beta\,\text{fin.}\,\psi$, hinc calculo rite fubducto pervenio ad hanc aequationem: $(b\,\text{fin.}\,(\alpha+\psi) - d\,\text{fin.}\,(\psi-\beta)) \sqrt{(c^2 - d^2\,\text{fin.}\,\beta^2)} = d\,\text{fin.}\,\beta\,(b\cos.\,(\beta+\psi) - d\cos.\,(\psi-\beta)) + c^2\,\text{fin.}\,\psi$ Q. E. I.

Coroll. I.

§. 394. Ex hac aequatione ftatim fequitur effe latus $BC = b =$

$$= \frac{d\,\text{fin.}(\psi-\beta)\sqrt{(c^2-d^2\,\text{fin}\,\beta^2)} - d^2\,\text{fin}\,\beta\cos.(\psi-\beta) + c^2\,\text{fin.}\,\psi}{\text{fin.}\,(\alpha+\psi)\sqrt{(c^2-d^2\,\text{fin.}\,\beta^2)} - d\,\text{fin.}\,\beta\cos.(\alpha+\psi)}$$

Sed pro latere AD obtinendo fit ad abbreviandum $\text{fin.}\,\beta = h$, $\text{fin.}\,\psi = m$, $b\,\text{fin.}(\alpha+\psi) - d\,\text{fin.}(\psi-\beta) = M$, $b\cos.\,(\alpha+\psi) - d\cos.\,(\psi-\beta) = N$, et factis fubftitutionibus erit aequatio abbreviata: $M\sqrt{(c^2 - d^2 h^2)} = Ndh + c^2 m$, ex qua operatione ad finem perducta obtinetur haec expreffio:

$$c = \frac{\sqrt{(-(M+2Ndhm) \pm M^2 + 4\,d\,h\,m\,(N - dhm))}}{m\sqrt{2}}$$

Pro latere DC obtinendo fit ad abbreviandum $\text{fin.}\,\beta = h$, $\text{fin.}\,\psi = k$, $\text{fin.}(\alpha+\psi) = m$, et $\cos.(\alpha+\psi) = n$, $\text{fin.}\,(\psi-\beta) = p$, $\cos.\,(\psi-\beta) = q$, factis fubftitutionibus, erit aequatio abbreviata: $(bm - dp)\sqrt{(c^2 - d^2h^2)} = dh\,(bn - dq) + c^2 k$, ex qua fumtis quadratis, facta transpofitione et concinnatione et brevitatis ergo etiam pofito $nq + mp = \cos.\,(\alpha+\beta) = r$, habetur haec aequatio quarti

 gradus

gradus: $d^4 - 2brd^3 + (b^2 - (\frac{2kq}{h} + \frac{p^2}{h^2})c^2)$
$d^2 + 2bc^2(\frac{nk}{h} + \frac{mp}{h^2})d + \frac{(c^2k^2 - b^2m^2)c^2}{h^2} = 0.$

Coroll. II.

§. 395. Quod si sinus et cosinus a se invicem separentur, et ad abbreviandum ponatur: $fin.\, \alpha = m$, $cos.\, \alpha = n$, $fin.\, \beta = h$, $cos.\, \beta = k$, et $\sqrt{(c^2 - d^2h^2)} = M$, pro obtinendo angulo A, operatione ad finem perducta, pervenio ad hanc aequationem: $tang.\, \psi = \frac{dh(bn - dk - M) - Mbm}{M(bn - dk) + dh(bm + dh) - c^2}$

Ad angulum ACB inveniendum sit ad abbreviandum ut prius $fin.\, \beta = h$, $fin.\, \psi = k$, $fin.\, \psi - \beta) = m$, $cos(\psi - \beta) = n$, et $\sqrt{(c^2 - d^2h^2)} = M$, $fin.\, (\alpha + \psi) = x$, et $cos.\, (\alpha + \psi) = \sqrt{(1 - x^2)}$, et facta substitutione et transpositione habetur haec aequatio abbreviata: $Mbx + d^2hn - Mdm - c^2k = dbh\sqrt{(1 - x^2)}$. Sit rursus ad abbreviandum aggregatum trium terminorum cognitorum $d^2hn - Mdm - c^2k = A$, ut fiat aequatio abbreviata: $Mbx + A = dbh\sqrt{(1 - x^2)}$, ex qua sumtis quadratis, absoluta reductione et concinnatione habetur haec aequatio:

$$x = \frac{-AMb \pm bdh\sqrt{(b^2c^2 - A^2)}}{b^2c^2} = fin.\, (\alpha + \psi)$$

et consequenter ipse angulus quaesitus erit:

$$\alpha = ang.\, fin. \frac{(-AMb \pm bdh\sqrt{(b^2c^2 - A^2)})}{b^2c^2} - \psi.$$

Schol.

Scholion I.

§. 396. In hoc problemate tres continentur caſus Tetragonometriae proprii, qui omnes in triangulo dextro obtinent, quando latus alterutrum vel angulus ACD quaeritur; horum unius, dum quaeritur latus AD, in corollario I. perfectam ſolutionem dedi, ſecundi caſus ſolutionem in eodem coroll. ad reſolutionem aequationis quarti gradus perduxi. Verum aequatio pro caſu tertio, quando quaeritur angulus ACD ad octavum gradum aſcendit et magna conſtat multitudine terminorum, quam igitur ut ad praxin parum utilem adferre, operae pretium non videbatur; tres autem hi caſus in Geometria practica, tria problemata nova ac utilia ſuppeditant; tres vero caſus caeteri utrique methodo et trigonometricae et tetragonometricae pariter ſubſunt, ſed ita tamen ut trigonometrice facilius et expeditius ſolvantur.

Coroll. III.

§. 397. Si anguli diagonaliter oppoſiti A et C ſint ſimul ſumti aequales duobus rectis ut prodeat trapezium circulo inſcriptibile; aequatio pro latere BC in hanc mutatur, eliminato angulo ψ,

$$b = \frac{d\,\text{ſin.}\,(\alpha + 2\beta)\,\sqrt{(c^2 - d^2\,\text{ſin.}\,\beta^2)}}{\text{ſin.}\,\beta\,\sqrt{(c^2 - d^2\,\text{ſin.}\,\beta^2)} - d\,\text{ſin.}\,\beta\,\cos.\,\beta}$$
$$+ \frac{d^2\,\text{ſin.}\,\beta\,\cos.\,(\alpha + 2\beta) + c^2\,\text{ſin.}\,(\alpha+\beta)}{\text{ſin.}\,\beta\,\sqrt{(c^2 - d^2\,\text{ſin.}\,\beta^2)} - d\,\text{ſin.}\,\beta\,\cos.\,\beta}$$

Verum in aequatione pro latere AD, erit: $b\,\text{ſin.}\,\beta - d\,\text{ſin.}\,(\alpha+2\beta) = M$, et $-b\cos.\,\beta + d\cos.\,(\alpha+2\beta) = N$, et $\text{ſin.}\,(\alpha+\beta) = m$, ſed in aequatione quarti

gradus

gradus pro latere CD, erit $k = fin.(\alpha+\beta)$, $m = fin.\beta = h$, $n = - cos.\beta$, $p = fin.(\alpha+2\beta)$, $q = - cos.(\alpha+2\beta)$ et $nq + mp = cos.\beta\, cos.(\alpha+2\beta) + fin.\beta\,(\alpha+2\beta) = cos.(\alpha+\beta) = r$, uti prius. Quare, fi $cot.\beta$ dicatur t; aequatio paulo fimplicius fcribitur hoc modo: $d^4 - 2brd^3 + \left(b^2 - \left(\frac{2kq}{h} + \frac{p^2}{h^2}\right)\right)$
$d^2 - 2bc^2\frac{(tk+p)}{h}d + \frac{(c^2k^2 - b^2)c^2}{h^2} = 0.$

Coroll. IV.

§. 398. Si angulus A ponatur rectus, vel etiam tribus rectis aequalis, cadente vertice A fupra diagonalem BD, intra vel extra triangulum BCD, et trapezio inverfo prodeunte; habetur pro latere BC haec aequatio:

$$b = \frac{d\, cos.\beta\, \surd(c^2 - d^2 fin.\beta^2) + d\, fin.\beta^2 \pm c^2}{cos.\alpha\, \surd(c^2 - d^2 fin.\beta^2) \mp d\, fin.\alpha\, fin.\beta},$$

fed in aequatione pro latere AD, fit: $m = \pm 1$, $M = \pm b\, cos.\alpha \mp d\, cos.\beta$, et $N = \pm b\, fin.\alpha \mp d\, fin.\beta$, ipfa vero aequatio paullo fimplicior prodit hoc modo:

$$c = \frac{\surd(-(M^2 - 2Ndh) \pm M\surd(M^2 - 4dh(N-dh)))}{m\surd 2}.$$

In aequatione quarti gradus fit: $k = \pm 1$, $m = \pm cos.\alpha$, $n = \mp fin.\alpha$, $p = \pm cos.\beta$, $q = \pm fin.\beta$, quare fi $cot.\beta$ dicatur t, aequatio fit paullo fimplicior hoc modo: $d^4 - 2brd^3 + (b^2 - (2+t^2)c^2)$
$d^2 - 2bc^2\left(\frac{n}{h} - \frac{mp}{h^2}\right)d^2 + \frac{(c^2 - b^2m^2)c^2}{h^2} = 0.$ In aequatione pro angulo ACB, fit etiam $k = \pm 1$, $m = \pm cos.\beta$, $n = \pm fin.\beta$, $fin.(\alpha+\psi) = \pm cos.\alpha$, et

et hanc ob rem fit: $cos.\alpha = \frac{\mp AMb \pm bdh \surd(b^2c^2 - A^2)}{b^2 c^2}$

et consequenter,

$$\alpha = ang.\ cos. \frac{\mp AMb \pm bdh \surd(b^2c^2 - A^2)}{b^2 c^2}.$$

Coroll. V.

§. 399. Si anguli ACB, ACD simul sumti, h. e. angulus C ponatur rectus vel etiam tribus rectis aequalis, cadente vertice C infra diagonalem BD intra vel extra triangulum DAB, et trapezio inverso prodeunte, pro latere BC habetur haec aequatio, β eliminato:

$$b = \frac{d\,cos.(\alpha+\psi)\surd(c^2 - d^2cos.\alpha^2) - d^2cos.\alpha\,sin.(\alpha+\psi) + c^2 sin.\psi}{sin.(\alpha+\psi)\surd(c^2 - d^2cos.\alpha^2) \pm d\,cos.\alpha\,cos.(\alpha+\psi)}$$

vel etiam,

$$b = \frac{d\surd(c^2 - d^2cos.\alpha^2) \pm d^2cos.\alpha\,tang.(\alpha+\psi)c^2 sin.\psi\,sec.(\alpha+\psi)}{tang.(\alpha+\psi)\surd(c^2 - d^2cos.\alpha^2) + d\,cos.\alpha}$$

Sed eliminato α, habetur haec expressio:

$$b = \frac{d\,sin.(\psi-\beta)\surd(c^2 - d^2sin.\beta^2) - d^2 sin.\beta\,cos.(\psi-\beta) + c^2 sin.\psi}{cos.(\psi-\beta)\surd(c^2 - d^2sin.\beta^2) \mp d\,sin.\beta\,sin.(\psi-\beta)}$$

vel etiam,

$$b = \frac{d\,tang.(\psi-\beta)\surd(c^2 - d^2sin.\beta^2) - d^2 sin.\beta + c^2 sin.\psi\,sec.(\psi-\beta)}{\surd(c - d^2 sin.\beta^2) \pm d\,sin.\beta\,tang.(\psi-\beta)}$$

At in aequatione pro latere AD fit: β eliminato, $M = b\,sin.(\alpha+\psi) \pm d\,cos(\alpha+\psi)$, sed α eliminato fit: $M = \pm\,cos.(\psi-\beta) - d\,sin.(\psi-\beta)$, in priore casu fit: $N = b\,cos.(\alpha+\psi) \mp d\,sin.(\alpha+\psi)$, in posteriore, $= \mp b\,sin.(\psi-\beta) - d\,cos.(\psi-\beta)$ in aequatione quarti gradus pro latere CD fit, eliminato β, $h = \pm cos.\alpha$, $p = \mp cos.(\alpha+\psi) = \mp n$, et $q = \pm sin.(\alpha+\psi) = \pm m$, et $r = 0$, hinc aequatio fit paullo simplicior

cior hujusmodi: $d^4 + \left(b^2 - \frac{c^2(n^2 \mp 2hkm)}{h^2}\right) d^2 - \frac{2bc^2 n(m+hk)d}{h} + \frac{(c^2k^2 - b^2m^2)i}{h^2} = o.$ Denique in aequatione pro angulo A, fit: $h = \pm n$, et $m = \mp k$ et aequatio paululum immutatur hoc modo:

$$tang.\, \psi = \frac{dh(\pm bh \mp dk - M) \mp Mbk}{M(\pm bh \mp dk) \pm dh(bk + dh) - c^2}.$$

Coroll. VI.

§. 400. Si angulus A ponatur aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus et trapezio in triangulum BCD abeunte: aequatio pro latere in hanc mutatur: $b = \frac{fin.\beta\,(d\sqrt{}(\sqrt{}(c^2 - d^2 fin.\beta^2) + d^2 cos.\beta)}{fin.\alpha \sqrt{}(c^2 - d^2 fin.\beta^2) + d\, fin.\beta\, cos.\alpha}$, in hac hypothesi cum sit $M = -b\, fin.\alpha - d\, fin.\beta$ et $N = -b\, cos.\alpha + d\, cos.\beta$; et aequatio quae primo prodiit in hanc mutatur: $(b\, fin.\alpha + d\, fin.\beta) \sqrt{}(c^2 - d^2 fin.\beta^2) = d\, fin.\beta\,(b\, cos.\alpha - d\, cos.\beta)$ et posito ad abbreviandum ut in coroll. 2. fit: $(bm + dh)\sqrt{}(c^2 - d^2h^2) = dh(bn - dk)$ ex qua obtinetur pro segmento AD lateris BD haec expressio: $c = \frac{d\, fin.\beta\, \sqrt{}(b^2 + d^2 - 2bd\, cos.(\alpha - \beta))}{b\, fin.a + d\, fin.\beta}.$

In aequatione quarti gradus pro latere CD fit: $m = -fin.\alpha$, $n = -cos.\alpha$, $p = fin.\beta = h$, $q = -cos.\beta$, $k = o$, $nq + mp = -cos.(\alpha + \beta) = r$, quare prodit aequatio quarti gradus simplicior:

$$d^4 - 2brd^3 + (b^2 - c^2)d^2 - \frac{2bc^2 md}{h} - \frac{b^2c^2m^2}{h^2} = o.$$

In expressione pro angulo ACB fit: $k = o$ $m =$

$m = h = fin.\beta$, $n = -cos.\beta$, $fin.(\alpha+\psi) = -fin.\alpha$, $A = dh(dn - M)$ quare haec prodit aequatio: $fin.\alpha = AMb \mp bdh \sqrt{(b^2c^2 - A^2)}$ et hinc ipfe angulus, $\alpha = ang.fin.\frac{(AMb \mp bdh\sqrt{(b^2c^2-A^2)})}{b^2c^2}$.

Scholion II.

§. 401. His aequationibus ultimi Corollarii in triangulo BCD nova quadam methodo inveniuntur latera BC, CD, fegmentum DE lateris BD, et angulus ECB, cum fi latus BC vel CD quaeratur, non dentur nifi latus, angulus adjacens et pars lateris tertii DE, fi quaeratur DE in triangulo partiali ECD, non dantur nifi duo, fcilicet angulus ECD et latus adjacens, fed in altero partiali EBC latus BC et angulus adjacens. Denique fi quaeratur angulus partialis ECB, in triangulo BCE non datur nifi latus adjacens; in altero partiali tria dantur; in totali duo integra latera, pars autem lateris tertii BD et pars anguli C. Verum evolutioni aequationis pro angulo ECD fuperfedeo, cum ad fextum gradum afcendat et magna conftet multitudine terminorum, latus autem CD tetragonometrice tantum invenitur.

Coroll. VII.

§. 402. Si angulus C ponatur aequalis duobus rectis, lateribus CB, CD, in diagonalem BD incidentibus, et trapezio in triangulum BAD abeunte, pro fegmento BE lateris BD prodit haec aequatio, β eliminato:

$$b = \frac{d\,tang.(\alpha+\psi)\sqrt{(c^2-d^2fin.\alpha^2)} + d^2cos.\alpha + c^2fin\psi fec.(\alpha+\psi)}{tang.(\alpha+\psi)\sqrt{(c^2-d^2fin.\alpha^2)} - d\,fin.\alpha}$$

vel

vel etiam haec altera, α eliminato:

$$b = \frac{d\,tang.(\psi-\beta)\sqrt{(c^2-d^2 fin.\beta^2)}-d^2 fin.\beta+c^2 fin.\psi fec.(\psi-\beta)}{tang.\psi-\beta\sqrt{(c^2-d^2 fin.\beta^2)}+d\,fin.\beta}$$

Sed in aequatione quadratica pro latere AD fit $M=-(b+d)\,fin.(\psi-\beta)$, eliminato α, at β eliminato fit: $M=(b+d)\,fin.(\alpha+\psi)$, in priore cafu fit: $N=-(b+d)\,cos.(\psi-\beta)$, et in pofteriore, $N=(b+d)\,cos.(\alpha+\psi)$, in aequatione quarti gradus pro latere ED fit: $p=-m$, $q=-n$, $r=-t$, et aequatio paululum mutatur hoc modo:

$$d^4+2\,b\,d^3+\left(b^2-\frac{c^2\,(2\,h\,k\,n+m^2)}{h^2}\right)d^2-\frac{2\,b\,c^2\,(m^2-n\,h\,k)}{h^2}d+\frac{(c^2\,k-b^2\,m^2)\,c^2}{h^2}=0.$$

Denique in expreffione pro angulo A fit $m=h$, et $n=-k$, quae igitur paululum immutata prodit hoc modo: $tang.\,\psi=\dfrac{d\,h((b+d)n-M(b+1))}{M(b+d)n+d\,h^2(b+d)-c^2}$.

Scholion III.

§. 403. Etiam his aequationibus nova quadam methodo inveniuntur quaefita, praefertim, fi quaeratur latus AD vel fegmentum ED, cum in his cafibus neque in triangulo totali ABD, neque partialibus ABE, AED dentur nifi duo, in caeteris tribus cafibus, dantur in totali duo, in partiali AED tria, in partiali ABE unum vel duo.

Scholion IV.

§. 404. In hypothefi figurae conftructae valet aequatio fupra pro trapezio data, quando unum trian-

triangulum a dextra, alterum a ſiniſtra diagonalis cadit, verum aequatio pro omni trapezio poſibili generalis eſt hujusmodi: $(b\,\sin.(\psi \pm \alpha) - b\,\sin.(\psi \mp \alpha)) \surd (c^2 - d^2 \sin.\beta^2) = \pm\, d \sin.\beta\, (b \cos.(\psi \pm \alpha) \mp d \cos.(\psi \mp \beta)) \pm c^2 \sin.\psi$, cujus omnia ſigna ſuperiora valent uno triangulo ad dextram, altero ad ſiniſtram cadente, ſi trapezium ſit partialiter inverſum, ita ut totum triangulum dextrum, ſive ad dextram, ſive ad ſiniſtram diagonalis cadat intra alterum; pro angulis in utroque aequationis membro valet ſignum affirmativum, et conſequenter prioris termini ſignum ſuperius, et poſterioris inferius in utroque aequationis membro, quod cancellis incluſum. Verum dextri membri cancellis incluſi valet ſignum negativum ſive inferius, ac ultimi termini ſolitarii ſuperius vel affirmativum. Si vero trapezium ita ſit inverſum, ut totum triangulum ſiniſtrum ſive ad dextram, ſive ad ſiniſtram diagonalis cadat intra dextram, valet in angulis ſignum negativum, et conſequenter in utroque membro prioris termini ſignum inferius, poſterioris ſuperius, ſed totius dextri membri cancellis incluſi ac ultimi termini ſolitarii valet ſignum inferius ſive negativum, in hoc itaque caſu valent omnia ſigna negativa. Quod ſi trapezium ſit totaliter inverſum, ita ut latus AD ſive ad dextram ſive ad ſiniſtram diagonalis cadat intra latus AB, et praeterea normalis ab angulo D in diagonalem demiſſa cadat intra figuram valent omnia ſigna ſecundi caſus, ſi vero perpendiculum cadat extra figuram, caeteris manentibus iisdem, valet ſignum inferius membri dextri cancellis incluſi, et poſterioris termini ejusdem, ut ſuperius, ultimi termini ſolitarii

litarii. Si trapezium ſit totaliter inverſum, latus vero AB ſive ad dextram ſive ad ſiniſtram cadat intra latus AD, et normalis ab angulo B intra figuram, prodit aequatio caſus tertii, in quo obtinent omnia ſigna negativa; ſi vero cadat extra figuram, caeteris manentibus iisdem ac in primo caſu, valet ſignum affirmativum termini poſterioris dextri membri cancellis incluſi et negativum termini ultimi ſolitarii. Caeterum attentione adhibita patet, quomodo ſigna in aequationibus immutanda ſint, ut ſolutio etiam ad trapezium ſive partialiter ſive totaliter inverſum applicetur.

Problema XXVIII.

Fig. XXVIII. §. 405. In figura quadrilatera propoſita $ABCD$ inter haec ſex: latus $BC=b$, $CD=d$, angulum $A=\psi$, $D=\varphi$, $ACB=\alpha$, et $ACD=\beta$ aequationem invenire.

Solutio.

Cum ſit ex reſolutione problematis XXVI. §.377. $\text{ſin.}\,CAD=\text{ſin.}(\beta+\varphi)\,\text{ſin.}\,CAB=-\text{ſin.}(\beta+\varphi+\psi)$ et $\text{ſin.}\,B=-\text{ſin.}\,(\alpha+\beta+\varphi+\psi)$, diagonalis $AC=\frac{d\,\text{ſin.}\,\varphi}{\text{ſin.}(\beta+\varphi)}$, in triangulo ſiniſtro formo hanc analogiam: $\text{ſin.}\,CAB : BC=\text{ſin.}\,B : AC$, h. e. $-\text{ſin.}(\beta+\varphi+\psi) : b=-\text{ſin.}\,(\alpha+\beta+\varphi+\psi) : \frac{d\,\text{ſin.}\,\varphi}{\text{ſin.}(\beta+\varphi)}$ et conſequenter habetur haec aequatio: $b\,\text{ſin.}\,(\beta+\varphi)\,\text{ſin.}\,(\alpha+\beta+\varphi+\psi)=d\,\text{ſin.}\,\varphi\ \text{ſin.}\,(\beta+\varphi+\psi)$, Q. E. I.

Coroll.

Coroll. I.

§. 406. Ex hac aequatione ſtatim incurrit in oculos eſſe: $b = \frac{d\,\sin.\,\varphi\,\sin.\,(\beta+\varphi+\psi)}{\sin.\,(\beta+\varphi)\,\sin.\,(\alpha+\beta+\varphi+\psi)}$, nec non $d = \frac{b\,\sin.\,(\beta+\varphi)\,\sin.\,(\alpha+\beta+\varphi+\psi)}{\sin.\,\varphi\,\sin.\,(\beta+\varphi+\psi)}$.

Coroll. II.

§. 407. Pro obtinendo angulo A habetur haec aequatio:

$$tang.\,\psi = \frac{tang.\,(\beta+\varphi)\,(d\,\sin.\,\varphi - b\,\sin.\,(\alpha+\beta+\varphi))}{b\,tang.\,(\beta+\varphi)\,cos.\,(\alpha+\beta+\varphi) - d\,\sin.\,\varphi}.$$

Si ad abbreviandum ponatur: $\beta = h$, $cos.\,\beta = k$, $\sin.\,(\beta+\psi) = m$, $cos.\,(\beta+\psi) = n$, $\sin.\,(\alpha+\beta+\psi) = p$, $cos.\,(\alpha+\beta+\psi) = q$, pro angulo D habetur haec aequatio: $tang.\,\varphi = \frac{b\,(h\,q + p\,k) - d\,m \pm}{2(d\,n - b\,k\,q)}$

$$\frac{\sqrt{((d\,m - (h\,q + p\,k)b^2) + 4\,b\,h\,p\,(d\,n - b\,k\,q))}}{2\,(d\,n - b\,k\,q)}$$

pro angulo $A\,C\,B$ habetur haec aequatio:

$$\alpha = ang.\,\sin.\,\frac{d\,\sin.\,\varphi\,\sin.\,(\beta+\varphi+\psi)}{b\,\sin.\,(\beta+\varphi)} - \beta - \varphi - \psi.$$

Pro inveniendo angulo ACD, ſeparatis a ſe invicem ſinibus, et poſito ad abbreviandum $\sin.\,\varphi = h$, $\sin.\,(\alpha+\psi) = m$, $cos.\,(\alpha+\psi) = n$, $\sin.\,\psi = p$, $cos.\,\psi = q$, $\sin.\,(\beta+\varphi) = x$, $cos.\,(\beta+\psi) = \sqrt{(1 - x^2)}$, factis ſubſtitutionibus et transpoſitione, proveniet haec aequatio abbreviata: $b\,n\,x^2 - d\,h\,q\,x = (d\,h\,p - b\,m\,x)\sqrt{(1 - x^2)}$, ex qua ſumtis quadratis, facta concinnatione transpoſitione et reductione,

ne, obtinetur haec aequatio quarti gradus, posito etiam $m\,p + n\,q == cos.\,\alpha == r$

$$x^4 - \frac{2dhrx^3}{b} + \frac{(d^2h^2 - b^2m^2)\,x^2}{b^2} + \frac{2\,d\,h\,m\,p\,x}{b} - \frac{d^2h^2p^2}{b^2} = 0.$$

In hac aequatione cum sit $x = ſin.(\beta + \varphi)$, φ autem in datorum numero; per extractionum radicis, dato $ſin.\,(\beta + \varphi)$, dabitur etiam ſumma angulorum $\beta + \varphi$, unde ſubtracto angulo dato φ dabitur etiam angulus quaeſitus β.

Scholion 1.

§. 408. Inter ſex caſus in aequatione contentos non dantur niſi duo Tetragonometriae proprii, qui obtinent, quando quaeruntur anguli D vel ACD, atque duo in Geometria practica ſuppeditant problemata nova, tum utilia, tum etiam pulchra. Prioris caſus ſolutionem dedi in ſecundo corollario per aequationem quadraticam, poſterioris ſolutionem ad aequationem quarti gradus deduxi; caeteri caſus quatuor utrique methodo et tetragonometricae et trigonometricae ſubſunt, quorum ſolutiones vulgaribus methodis trigonometricis fere palmam praeripiunt.

Coroll. III.

§. 409. Si anguli A et D, ſint ſimul ſumti duobus rectis aequales, prodeunte trapezio parallelarum baſium AB, DC aequatio pro latere BC in hanc mutatur: $b = \frac{d\,ſin.\,\varphi\,ſin.\,\beta}{ſin.(\alpha + \beta)\,ſin.(\beta + \varphi)}$

et

et consequenter $CD = d = \frac{b\,\text{fin.}(\alpha+\beta)\,\text{fin.}(\beta+\varphi)}{\text{fin.}\,\varphi\,\text{fin.}\,\beta}$ verum expressio pro angulo ACB, in hanc mutatur: $\alpha = \text{ang.fin.}\frac{d\,\text{fin.}\,\varphi\,\text{fin.}\,\beta}{b\,\text{fin.}\,(\beta+\varphi)} - \beta$ sed in aequatione quarti gradus, pro angulo ACD fit $h=p$, si vero anguli C et D, ponantur simul sumti duobus rectis aequales, prodeunte trapezio parallelarum basium AD, BC habetur pro latere BC, α elimimato, haec expressio: $b = \frac{-d\,\text{fin.}\,\varphi\,\text{fin.}\,(\beta+\varphi+\psi)}{\text{fin.}(\beta+\varphi)\,\text{fin.}\,\psi.}$, verum β eliminato habetur: $b = \frac{d\,\text{fin.}\,\varphi\,\text{fin.}\,(\psi-\alpha)}{\text{fin.}\,\alpha\,\text{fin.}\,\psi}$, et denique φ eliminato haec expressio: $b = \frac{d\,\text{fin.}(\alpha+\beta)\,\text{fin.}(\psi-\alpha)}{\text{fin.}\,\alpha\,\text{fin.}\,\psi}$ hinc erit latus $CD = d = \frac{-b\,\text{fin.}\,(\beta+\varphi)\,\text{fin.}\,\psi}{\text{fin.}\,\varphi\,\text{fin.}(\beta+\varphi+\psi)}$ $= \frac{b\,\text{fin.}\,\alpha\,\text{fin.}\,\psi}{\text{fin.}\,\varphi\,\text{fin.}\,(\psi-\alpha)} = \frac{b\,\text{fin.}\,\alpha\,\text{fin.}\,\psi}{\text{fin.}\,(\alpha+\beta)\,\text{fin.}(\psi-\alpha)}$. Pro angulo A habetur α eliminato haec expressio: $\text{tang.}\,\psi = \frac{d\,\text{tang.}\,(\beta+\varphi)\,\text{fin.}\,\varphi}{b\,\text{tang.}\,(\beta+\varphi) + d\,\text{fin.}\,\varphi}$. Si vero β eliminetur, haec obtinetur: $\text{tang.}\,\psi = \frac{d\,\text{tang.}\,\alpha\,\text{fin.}\,\varphi}{d\,\text{fin.}\,\varphi - b\,\text{tang.}\,\alpha}$ denique φ eliminato: $\text{tang.}\,\psi = \frac{d\,\text{tang.}\,\alpha\,\text{fin.}\,(\alpha+\beta)}{d\,\text{fin.}(\alpha+\beta) - b\,\text{tang.}\,\alpha}$.

Coroll. IV.

§. 410. Si anguli diagonaliter oppositi A et C sint simul sumti aequales duobus rectis, ut prodeat trapezium circulo inscriptibile, fit pro latere BC, α eliminato: $b = -d\,fin.(\beta+\phi+\psi) : fin.(\beta+\phi)$, β vero eliminato fit: $b = d\,fin.(\phi-\alpha) : fin.(\phi-\alpha-\psi)$ et denique ψ eliminato $b = d\,fin.(\phi-\alpha) : fin.(\beta+\phi)$; in primo igitur casu fit: $d = b\,fin.(\beta+\phi) : fin.(\beta+\phi+\psi)$, at in secundo, $d = b\,fin.(\phi-\alpha-\psi) : fin.(\phi-\alpha)$, et in tertio, $d = b\,fin.(\beta+\phi) : fin.(\phi-\alpha)$; verum aequatio pro angulo D, in hanc abit, eliminato α, $tang.\,\phi = \dfrac{-(b\,fin.\,\beta + d\,fin.\,(\beta+\psi))}{b\,cos.\,\beta + d\,cos.(\beta+\psi)}$, ψ autem eliminato, habetur: $tang.\,\phi = \dfrac{b\,fin.\,\beta + d\,fin.\,\alpha}{d\,cos.\,\alpha - b\,cos.\,\beta}$ et denique β eliminato, fit:

$$tang.\,\phi = \frac{b\,fin.(\alpha+\psi) + d\,fin.\,\alpha}{d\,cos.\,\alpha + b\,cos.(\alpha+\psi)}.$$

Coroll. V.

§. 411. Si angulus A sit rectus, vel etiam duobus rectis, aequalis cadente vertice A supra diagonalem BD, intra vel extra triangulum BCD, et trapezio inverso prodeunte, aequatio pro latere BC in hanc mutatur: $b = \dfrac{d\,fin.\,\phi\,cot.(\beta+\phi)}{cos.(\alpha+\beta+\phi)}$ et consequenter $d = \dfrac{b\,cos.(\alpha+\beta+\phi)}{fin.\,\phi\,cot.(\beta+\phi)}$, expressio pro angulo ACB in hanc mutatur: $\alpha = ang.\,cos.\dfrac{d\,fin.\,\phi\,cot.(\beta+\phi)}{b} - \beta - \phi$. In expressio-

preſſione tangentis pro angulo D ſit $m = \pm k$, $n = \mp h$, $p = \pm \cos.(\alpha+\beta)$, $q = \mp \sin.(\alpha+\beta)$, quare ſi pro $tang.\,\phi$ ſcribatur brevitatis ergo t, erit

$$t = \frac{\pm b(pk-hq) \mp dk \pm \surd((dk \mp b(hq-pk))^2}{\mp 2(dh-bkq)}$$

$$\frac{\mp 4bhp(dh-bkq))}{\mp 2(dh-bkq)}$$ in qua aequatione valent omnia ſigna ſuperiore tam in numeratore quam in denominatore pro poſitione anguli recti, inferiora pro poſitione anguli A tribus rectis aequali. In aequatione quarti gradus pro angulo ACD, fit $m = \pm r$, $n = \mp \sin.\alpha$; $\sin.\psi = \pm 1 = p$, $\sin.(\beta+\phi) = x = \pm \cos.\beta$, et conſequenter ipſa aequatio:

$$x^4 + \frac{2dhmx^3}{b} + \frac{(d^2h^2-b^2m^2)x^2}{b^2} + \frac{2dhmx}{b} - \frac{d^2h^2}{b^2} = 0.$$

Coroll. VI.

§. 412. Si angulus D ſit rectus vel tribus rectis aequalis, cadente vertice D ad ſiniſtram diagonalis intra vel extra triangulum ABC, et trapezio inverſo prodeunte, fit aequatio pro latere BC, $b = d\cos.(\beta+\psi) : \cos.\beta \cos.(\alpha+\beta+\psi)$ et pro latere CD, $d = b\cos.\beta \cos.(\alpha+\beta+\psi) : \cos.(\beta+\psi)$. Verum expreſſio pro angulo A in hanc mutatur: $tang.\,\psi = \frac{\mp \cot.\beta(d+b\cos.(\alpha+\beta))}{b\cot.\beta \sin.(\alpha+\beta)-d}$. Expreſſio pro angulo ACB in hanc abit: $\alpha = ang.\cos.\frac{d\cos.(\beta+\psi)}{b\cos.\beta} - \beta - \psi$. In aequatione quarti gradus pro angulo ACD, fit $h = \pm 1$, $x = \pm \cos.\beta$, et

et aequatio fit paullo ſimplicior talis:

$$x^4 - \frac{2drx^3}{b} + \frac{(d^2 - b^2m^2)x^2}{b^2} + \frac{2dmpx}{b} - \frac{d^2p^2}{b^2} = 0.$$

Coroll. VII.

§. 413. Si angulus C ſit rectus vel etiam tribus rectis aequalis, cadente vertice C, infra diagonalem BD, intra vel extra triangulum BAD, prodeunte trapezio inverſo, habetur pro latere BC haec aequatio α eliminato: $b = \frac{\pm d\,\text{ſin.}\,\phi\,\text{ſin.}\,(\beta+\phi+\psi)}{\text{ſin.}\,(\beta+\phi)\,\cos.\,(\phi+\psi)}$, ſed β eliminato habetur haec: $b = \frac{\pm d\,\text{ſin.}\,\phi\,\cos.\,(\phi+\psi-\alpha)}{\cos.\,(\phi-\alpha)\,\cos.\,(\phi+\psi)}$; in priore caſu eſt latus $CD = d = \frac{\pm b\,\text{ſin.}(\beta+\phi)\cos.(\phi+\psi)}{\text{ſin.}\,\phi\,\text{ſin.}\,(\beta+\phi+\psi)}$ in poſteriore $d = \frac{\pm b\cos.(\phi-\alpha)\cos.(\phi+\psi)}{\text{ſin.}\,\phi\cos.(\phi+\psi-\alpha)}$. Expreſſio pro angulo A in hanc mutatur: $\text{tang.}\,\psi = \frac{\text{tang.}\,(\beta+\phi)\,(b\cot.\,\phi \mp d)}{\text{tang.}\,(\beta+\phi) \pm d}$ ubi ſigna ſuperiora valent pro uno recto, inferiora pro tribus rectis. In aequatione quadratica pro angulo D fit $p = \pm \cos.\,\psi$, $q = \mp\,\text{ſin.}\,\psi$, verum aequatio paululum immutatur hoc modo:

$$\text{tang.}\,\phi = \frac{\pm b(pk-hq) - dm \pm \sqrt{((dm \mp b(hq-pk))^2}}{2(dn \pm bkq)}$$

$$\frac{\pm 4bhp\,(bkq \pm dn))}{2(dn \pm bkq)}$$

ubi pro poſitione recti valent ſigna ſuperiora, inferiora pro tribus rectis.

Coroll.

Coroll. VIII.

§. 414. Si angulus A ſit aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus, et trapezio in triangulum BCD abeunte; ſit latus $BC=b=\frac{d\ fin.\ \phi}{fin.\ (\alpha+\beta+\phi)}$, et in triangulo BCD prodit analogia trigonometrica $b:\ d = fin.\ \phi :\ fin.\ (\alpha+\beta+\phi)$; Si vero angulus C ponatur aequalis duobus rectis; lateribus CB, CD in diagonalem BD incidentibus et trapezio in triangulum ABD abeunte, ſit latus $BE=b=\frac{-d\ fin.\ \phi\ fin.\ (\beta+\phi+\psi)}{fin.\ (\beta+\phi)\ fin.\ (\phi+\psi)}$, unde prodit haec analogia $b : \frac{d\ fin.\ \phi}{fin.(\beta+\phi)} = fin.\ (\beta+\phi+\psi) :$ $-fin.\ (\phi+\psi)$. Priorem analogiam per Trigonometriam, poſteriorem per modum, quo ad aequationem perventum fuit, prodire debere conſtat, quibus igitur aequatio verificatur.

Scholion II.

§. 415. Si dextrum triangulum cadat intra ſiniſtrum, ut trapezium ſiat inverſum ſive ſit partialiter ſive totaliter inverſum, valet haec aequatio: $b\ fin.\ (\beta+\phi)\ fin.\ (\beta+\phi-\alpha-\psi) = d\ fin.\ \phi$ $fin.(\beta+\phi-\psi)$; quod ſi vero ſiniſtrum intra dextrum cadat, ut trapezium ſiat ſive partialiter ſive totaliter inverſum, valet haec aequatio: $b\ fin.(\beta+\phi)$ $fin.\ (\phi+\psi+\beta-\alpha) = d\ fin.\ \phi\ fin.\ (\beta+\phi+\psi)$, quare generalis aequatio pro omni trapezio erit talis: $b\ fin.(\beta+\phi)\ fin.(\beta+\phi\pm\psi\pm\alpha) = d\ fin.\phi\ fin.(\beta+\phi\pm\psi)$. Ex dictis igitur apparet, ſi detur ſpecies trapezii, quaenam ſigna ſint uſurpanda, et quomodo in aequationibus pro lateribus et angulis mutanda.

CAPUT VIII.

Continens tria problemata ſecundae claſſis particularis, ſub poſteriore principali contentae.

Problema XXIX.

Fig. XXIX. §. 416. In figura quadrilatera propoſita $ABCD$ inter haec ſex: latus $AB=a$ $AD=c$, angulum $A=\psi$, $D=\varphi$, $ACB=\alpha$, et $ACD=\beta$, aequationem invenire

Solutio.

Cum ſit e ſolutione problematis XXVI. (§. 377.) diagonalis $AC=\frac{c\ fin.\ \varphi}{fin.\ \beta}$, et per antecedentia $fin.\ B=-fin.\ (\alpha+\beta+\varphi+\psi)$; in triangulo ſiniſtro ſtatim pervenio ad hanc analogiam: $AB: fin.\ ACB=AC: fin.\ B$, h. e. ſubſtitutis ſymbolis $a: fin.\ \alpha=\frac{c\ fin.\ \varphi}{fin.\ \beta}: -fin.\ (\alpha+\beta+\varphi+\psi)$, unde fit aequatio talis: $c\ fin.\ \alpha\ fin.\ \varphi = -a\ fin.\ \beta\ fin.\ (\alpha+\beta+\varphi+\psi)$. Q. E. I.

Coroll. I.

§. 417. Ex hac aequatione ſtatim in oculos incurrit eſſe latus $AB=a=\frac{-c\ fin.\ \alpha\ fin.\ \varphi}{fin.\ \beta\ fin.\ (\alpha+\beta+\varphi+\psi)}$ et latus $AD=c=\frac{-a\ fin.\ \beta\ fin.\ (\alpha+\beta+\varphi+\psi)}{fin.\ \alpha\ fin.\ \varphi}$.

Coroll.

Coroll. II.

§. 418. Cum fit $fin.(\alpha+\beta+\varphi+\psi)=\frac{-c\,fin.\,\alpha\,fin.\,\varphi}{a\,fin.\,\beta}$, fequitur effe ipfum angulum $\psi = ang.\,fin.\frac{-c\,fin.\,\alpha\,fin.\,\varphi}{a\,fin.\,\beta} - \beta - \varphi - \alpha$. Verum fatis brevis et concinna habetur aequatio, finus a fe invicem feparando: facta igitur feparatione et pofito ad abbreviandum $fin.\,\alpha = h$, $fin.\,\beta = k$, $fin.\,\varphi = m$, $fin.(\alpha+\beta+\varphi) = p$, $cos.(\alpha+\beta+\varphi) = q$, $fin.\,\psi = x$, $cos.\,\psi = \sqrt{(1-x)}$ et operatione ad finem perducta habetur haec aequatio quadratica:

$$x = \frac{-c\,h\,m\,q \pm p\sqrt{(a^2k^2 - c^2h^2m^2)}}{a\,k} = fin.\,\psi.$$

Coroll. III.

§. 419. Pro angulo ACB obtinendo habetur haec aequatio:

$$tang.\,\alpha = \frac{-a\,fin.\,\beta\,fin.(\beta+\varphi+\psi)}{c\,fin.\,\varphi + a\,fin.\,\beta\,cos.(\beta+\varphi+\psi)}$$

vel etiam haec:

$$tang.\,\alpha = \frac{-a\,fin.\,\beta\,tang.(\beta+\varphi+\psi)}{a\,fin.\,\beta + c\,fin.\,\varphi\,fec.(\beta+\varphi+\psi)};$$

et pro angulo D habetur haec:

$$tang.\,\varphi = \frac{-a\,fin.\,\beta\,fin.(\alpha+\beta+\psi)}{c\,fin.\,\alpha + a\,fin.\,\beta\,cos.(\alpha+\beta+\psi)}$$

vel etiam haec altera:

$$tang.\,\varphi = \frac{-a\,fin.\,\beta\,tang.(\alpha+\beta+\psi)}{a\,fin.\,\beta + c\,fin.\,\alpha\,fec.(\alpha+\beta+\psi)}$$

hinc erit ipfe angulus:

$$\alpha = ang.\,tang.\frac{-a\,fin.\,\beta\,tang.(\alpha+\beta+\psi)}{a\,fin.\,\beta + c\,fin.\,\varphi\,fec.(\beta+\varphi+\psi)}$$ et an-

 gulus

gulus $D = \varphi\, ang.\, tang. \frac{-a\, fin.\, \beta\, tang.\, (\alpha+\beta+\varphi)}{a\, fin.\, \beta + c\, fin.\, \alpha\, fec.\, (\beta+\varphi+\psi)}$.
Verum ad obtinendum angulum $A\ C\ D$ aequationem in hanc formam transmuto:

$$\frac{2c\, fin.\, \alpha\, fin.\, \varphi + a\, cos.\, (\alpha+\varphi+\psi)}{a} = cos.\, (\alpha+2\beta+\varphi+\psi),$$

unde ad angulum ipsum descendendo obtineo

$$\alpha+2\beta+\varphi+\psi = ang.\, cos. \frac{2c\, fin.\, \alpha\, fin.\, \varphi + a\, cos.\, (\alpha+\varphi+\psi)}{a}$$

et consequenter,

$$\beta = \tfrac{1}{2}\, ang.\, cos. \frac{2c\, fin.\, \alpha\, fin.\, \varphi + a\, cos.\, (\alpha+\varphi+\psi)}{a} - \tfrac{1}{2}\alpha - \tfrac{1}{2}\varphi - \tfrac{1}{2}\psi.$$

Coroll. IV.

§. 420. Si sinistrum aequationis membrum multiplicetur per $fin.\, \beta^2 + cos.\, \beta^2 = 1$ ad aequationem respectu anguli β homogeneam reddendam, sinibus a se invicem separatis, et ad abbreviandum posito $fin.\, \alpha = h$, $fin.\, \varphi = k$, $fin.\, (\alpha+\varphi+\psi) = m$, $cos.\, (\alpha+\varphi+\psi) = n$, $tang.\, \beta = x$; facta substitutione et transpositione, habetur haec aequatio abbreviata: $(an+chk)x^2 + amx = -chk$, ex qua reducta obtinetur: $x = \frac{-am \pm \sqrt{(a^2m^2 - 4chk(an+chk)}}{2(an+chk)}$.

Scholion I.

§. 421. Inter sex casus hujus problematis, qui in aequatione continentur, non dantur nisi duo Tetragonometriae proprii, qui obtinent, quando angulus D, vel ACD quaeritur, et in Geometria practica suppeditant problemata nova,

tam

tam utilia quam ſatis pulchra, horum caſuum unius unam, alterius duas dedi ſolutiones. Caeteri quatuor caſus utrique methodo et trigonometricae et tetragonometricae ſubſunt, omnes autem evolvi, quia tum ex aequatione breviter et expedite ſequntur, tum etiam quia ſolutiones vulgaribus trigonometricis parum vel nihil cedunt.

Coroll. V.

§. 422. Si anguli A et D, ſint ſimul ſumti aequales duobus rectis, prodeunte trapezio parallelarum baſium AB, CD, aequatio pro latere AB in hanc mutatur, ψ eliminato: $a = \frac{c\, fin.\, \alpha\, fin.\, \phi}{fin.\, \beta\, fin.\, (\alpha+\beta)} = \frac{c\, fin.\, \alpha\, fin.\, \psi}{fin.\, \beta\, fin.\, (\beta+\alpha)}$, ϕ eliminato; et pro latere AD fit: $c = \frac{a\, fin.\, \beta\, fin.\, (\alpha+\beta)}{fin.\, \alpha\, fin.\, \phi} = \frac{a\, fin.\, \beta\, fin.\, (\alpha+\beta)}{fin.\, \alpha\, fin.\, \psi}$ Verum aequatio pro angulo ACB, in hanc mutatur: $tang.\, \alpha = \frac{a\, fin.\, \beta^2}{c\, fin.\, \phi - a\, fin.\, \beta\, cos.\, \beta} = \frac{a\, tang.\, \beta^2}{c\, fin.\, \phi\, fec.\, \beta^2 - a\, tang.\, \beta}$ vel etiam angulo ϕ eliminato habetur: $tang\, \alpha = \frac{a\, fin.\, \beta^2}{c\, fin.\, \psi - a\, fin.\, \beta\, cos.\, \beta} = \frac{a\, tang.\, \beta^2}{c\, fin.\, \psi\, fec.\, \beta^2 - a\, tang.\, \beta}$; aequatio prior pro angulo ACD in hanc mutatur: $\beta = \frac{1}{2}\, ang.\, cos. \frac{-2c\, fin.\, \alpha\, fin.\, \phi + a\, cos.\, \alpha}{a} - \frac{\alpha}{2}$ ſed in aequatione quadratica coroll. IV. fit, $m = -h$, et $n = -cos.\, \alpha$, et conſequenter aequatio paululum in hanc mutatur: $x = \frac{ah \pm \surd(a^2h^2 - 4chk\, (chk - an)}{2\, (chk - an)}$

Quod

Quod ſi anguli C et D ſimul ſumti ſint aequales duobus rectis, prodeunte trapezio parallelarum baſium CB, AD, fit pro latere AB, $a = \frac{c\, fin.\, \alpha\, fin.\, \varphi}{fin.\, \beta\, fin.\, \psi}$ et hinc pro latere AD $c = \frac{a\, fin.\, \beta\, fin.\, \psi}{fin.\, \alpha\, fin.\, \varphi}$, hinc pro angulo A habetur: $fin.\, \psi = \frac{c\, fin.\, \alpha\, fin.\, \varphi}{a\, fin.\, \beta}$ et $\psi = ang.\, fin.\, \frac{c\, fin.\, \alpha\, fin.\, \varphi}{a\, fin.\, \beta}$. Si anguli diagonaliter oppoſiti ſint aequales duobus rectis ut prodeat trapezium circulo inſcriptibile, fit pro latere $AB = a = \frac{c\, fin.\, \alpha}{fin.\, \beta}$.

Coroll. VI.

§. 423. Si angulus A ſit rectus vel etiam tribus rectis aequalis, cadente vertice A ſupra diagonalem BD intra triangulum BCD, vel etiam extra, ut prodeat trapezium inverſum, aequatio pro latere AB in hanc mutatur: $a = \frac{\mp c\, fin.\, \alpha\, fin.\, \varphi}{fin.\, \beta\, cos.(\alpha+\beta+\varphi)}$, et pro latere AD fit $c = \frac{\mp a\, fin.\, \beta\, cos.(\alpha+\beta+\varphi)}{fin.\, \alpha\, fin.\, \varphi}$, pro angulo ACB fit $tang.\, \alpha = \frac{\mp a\, fin.\, \beta\, cos.\, (\beta+\varphi)}{c\, fin.\, \varphi \mp fin.\, \beta\, fin.(\beta+\varphi)}$ vel etiam $tang.\, \alpha = \frac{\mp a\, fin.\, \beta}{c\, fin.\, \varphi\, fec.(\beta+\varphi) \mp a\, fin.\, \beta\, tang.(\beta+\varphi)}$, verum pro angulo D habetur: $tang.\, \varphi = \frac{\mp a\, fin.\, \beta\, cos.(\alpha+\beta)}{c\, fin.\, \alpha \mp a\, fin.\, \beta\, fin.(\alpha+\beta)}$ vel etiam

tang.

$tang.\, \varphi = \frac{\mp a\, fin.\, \beta}{c\, fin.\, \alpha\, fec.\, (\alpha+\beta) \mp a\, fin.\, \beta\, tang.\, (\alpha+\beta)}$;
pro angulo ACD aequatio ita mutatur, ut fiat
$\beta = \frac{1}{2}\, ang.\, fin. \frac{\mp 2c\, fin.\, \alpha\, fin.\, \varphi + a\, fin.\, (\alpha+\varphi)}{a} - \frac{\alpha - \varphi}{2}$;
in aequatione quadratica fit, $m = \pm\, cos.\, (\alpha+\varphi)$ et $n = \mp\, fin.\, (\alpha+\varphi)$, et consequenter aequatio in hanc mutatur: $x = \frac{\mp\, am \pm \sqrt{(a^2m^2 - 4chk(chk \pm an))}}{2(chk \mp an)}$.

Coroll. VII.

§. 424. Si angulus D fit rectus vel tribus rectis aequalis, cadente vertice D ad finistram diagonalis AC, prodeunte trapezio inverso; aequatio pro latere AB in hanc mutatur:
$a = \frac{-c\, fin.\, \alpha}{fin.\, \beta\, cos.\, (\alpha+\beta+\psi)}$ et pro latere AD
$c = \frac{-a\, fin.\, \beta\, cos.\, (\alpha+\beta+\psi)}{fin.\, \alpha}$ et pro angulo A fit
$\psi = ang.\, cos. \frac{-fin.\, \alpha}{a\, fin.\, \beta} - \alpha - \beta$. Sed in aequatione quadratica fit, $m = \pm 1$, $p = \pm\, cos.\, (\alpha+\beta)$, $q = \mp\, fin.\, (\alpha+\beta)$, et ipsa aequatio in hanc abit $x = \frac{chq \pm p\sqrt{(a^2k^2 - c^2h^2)}}{ak}$, pro angulo AcB fit,
$tang.\, \alpha = \frac{a\, fin.\, \beta}{a\, fin.\, \beta\, tang.\, (\beta+\psi) - c\, fec.\, (\beta+\psi)}$;
denique pro angulo ACD habetur:
$\beta = \frac{1}{2}\, ang.\, fin. \frac{-2c\, fin.\, \alpha + a\, fin.\, (\alpha+\psi)}{a} - \alpha - \psi$.

Sed

Sed in aequatione quadratica fit $k = \pm 1$, $m = \pm \cos.(\alpha+\psi)$, $n = \mp \sin.(\alpha+\psi)$ et aequatio in hanc mutatur: $x = \frac{-am \pm \surd(a^2m^2 - 4ch(ch-an))}{2(ch-an)}$.

Si angulus C fit rectus vel etiam duobus rectis aequalis, cadente vertice C infra diagonalem BD, et trapezio inverso prodeunte; aequatio pro latere AB in hanc mutatur:

$a = \frac{\pm c \sin. \alpha \sin. \phi}{\sin. \beta \cos. (\phi+\psi)}$, et pro latere AD fit, $c = \frac{\mp a \sin. \beta \cos. (\phi+\psi)}{\sin. \alpha \sin. \phi}$, pro angulo D,

$tang. \phi = \frac{a \sin. \beta}{a \sin. \beta \; tang. \psi \pm c \sin. \alpha \sec. \psi}$,

denique pro angulo A habetur:

$\psi = ang. \cos. \frac{\mp c \sin. \alpha \sin. \phi}{\alpha \sin. \beta} - \phi$. Sed in aequatione quadratica fit, $p = \pm \cos. \phi$, $q = \mp \sin. \phi = \mp m$, quae igitur in hanc mutatur:

$x = \frac{\pm c m^2 \pm p \surd(a^2k^2 - c^2h^2m^2)}{ak}$. Si denique angulus B ponatur rectus vel etiam tribus rectis aequalis, cadente vertice B ad dextram diagonalis AC et trapezio inverso prodeunte, fit pro latere AB $a = \frac{\mp c \sin. \alpha \sin. \phi}{\sin. \beta}$, et hinc pro latere AD $c = \frac{\mp a \sin. \beta}{\sin. \alpha \sin. \phi}$.

Coroll.

Coroll. VIII.

§. 425. Si angulus A ſit aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus, et trapezio in triangulum BCD abeunte, aequatio pro latere EB in hanc mutatur: $a = \frac{c\, ſin.\, \alpha\, ſin.\, \varphi}{ſin.\, \beta\, ſin.\, (\alpha+\beta+\varphi)}$ et conſequenter fit latus $ED = c = \frac{a\, ſin.\, \beta\, ſin.\, (\alpha+\beta+\varphi)}{ſin.\, \alpha\, ſin.\, \varphi}$; pro angulo ACB fit, $tang.\, \alpha = \frac{a\, ſin\, \beta\, ſin.\, (\beta+\varphi)}{c\, ſin.\, \varphi - a\, ſin.\, \beta\, cos.(\beta+\varphi)}$ vel etiam $tang.\, \alpha = \frac{a\, ſin.\, \beta\, tang.\, (\beta+\varphi)}{c\, ſin.\, \varphi\, ſec.\, (\beta+\varphi) - a\, ſin.\, \beta}$; pro angulo D habetur: $tang.\, \varphi = \frac{a\, ſin.\, \beta\, ſin.\, (\alpha+\beta)}{c\, ſin.\, \alpha - a\, ſin.\, \beta\, cos.(\alpha+\beta)}$ vel etiam $tang.\, \varphi = \frac{a\, ſin.\, \beta\, tang.\, (\alpha+\beta)}{c\, ſin.\, \alpha\, ſec.\, (\alpha+\beta) - a\, ſin.\, \beta}$, haec autem aequatio pro angulo D, ut et proxime ſequens pro angulo ACD, duo nova ad triangula pertinentia ſolvunt problemata, quae trigonometrice non ſolvuntur. Pro angulo ACD habetur haec expreſſio:

$$\beta = \tfrac{1}{2}\, ang.\, cos.\frac{-2c\, ſin.\, \alpha\, ſin.\, \varphi + a\, cos.(\alpha+\varphi)}{a} - \frac{\alpha-\varphi}{2}$$

ſed in aequatione quadratica quarti corollarii fit, $m = -ſin.\, (\alpha+\varphi)$, $n = -cos.\, (\alpha+\varphi)$ et aequatio quadratica in hanc paululum mutatur: $x = \frac{am \pm \sqrt{(a^2 m^2 - 4chk\,(chk - an))}}{2(chk - an)}$. Si angulus C ponatur aequalis duobus rectis, lateribus

CB, CD in diagonalem BD, incidentibus; aequatio pro latere EB in hanc mutatur: $\alpha = \frac{c\, \text{fin.}\, \alpha\, \text{fin.}\, \phi}{\text{fin.}\, \beta\, \text{fin.}\, (\phi + \psi)}$ et hinc $c = \frac{a\, \text{fin.}\, \beta\, \text{fin.}\, (\phi + \psi)}{\text{fin.}\, \alpha\, \text{fin.}\, \phi}$, vel etiam accuratius, cum fit $\text{fin.}\, \alpha = \text{fin.}\, \beta$, erit $a = \frac{c\, \text{fin.}\, \phi}{\text{fin.}\, (\phi + \psi)}$ et $c = \frac{a\, \text{fin.}\, (\phi + \psi)}{\text{fin.}\, \phi}$, unde fequitur haec analogia trigonometrica $a : \text{fin.}\, \phi = c : \text{fin.}\, (\phi + \psi)$ fit igitur angulus $\psi = \text{ang. fin.} \frac{c\, \text{fin.}\, \phi}{a} - \phi$;

verum aequatio quadratica, cum fit ex hypothefi $h = k$, $p = -\text{fin.}\, \phi = -m$, $q = -\text{cos.}\, \phi$, in hanc mutatur:

$$x = \frac{m(cq \mp \surd(a^2 - c^2 m^2))}{a} = \frac{\text{fin}\, \phi (c\, \text{cos.}\, \phi \mp \surd(a^2 - c^2\, \text{fin}\, \phi^2))}{a};$$

valoribus reftitutis pro angulo D habetur haec expreffio: $\text{tang.}\, \phi = \frac{a\, \text{fin.}\, \beta\, \text{fin.}\, \psi}{c\, \text{fin.}\, \alpha - a\, \text{fin.}\, \beta \text{cos.}\, \psi}$, vel etiam haec: $\text{tang.}\, \phi = \frac{a\, \text{fin.}\, \beta\, \text{tang.}\, \psi}{c\, \text{fin.}\, \alpha\, \text{fec.}\, \psi - a\, \text{fin.}\, \beta} = \frac{a\, \text{tang.}\, \psi}{c\, \text{fec.}\, \psi - a}$.

Scholion II.

§. 426. Si trapezium fit five partialiter five totaliter inverfum, ut latus AD cadat intra latus AB, valet haec aequatio: $c\, \text{fin.}\, \alpha\, \text{fin.}\, \phi = a\, \text{fin.}\, \beta\, \text{fin.}\, (\beta + \phi - \alpha - \psi)$. Si vero fit five partialiter five totaliter inverfum, ut latus AB cadat intra latus AD, valet haec aequatio: $c\, \text{fin.}\, \alpha\, \text{fin.}\, \phi = a\, \text{fin.}\, \beta\, \text{fin.}\, (\beta + \phi + \psi - \alpha)$; aequatio igitur generalis haec eft: $c\, \text{fin.}\, \alpha\, \text{fin.}\, \phi = \mp a\, \text{fin.}\, \beta\, \text{fin.}\, (\beta + \phi \pm \psi \pm \alpha)$.

Proble-

Problema XXX.

§. 427. In figura quadrilatera proposita $ABCD$ inter haec sex: latus $AB=a$, $AD=c$, $CD=d$, angulum $A=\psi$, $ACB=\alpha$, $ACD=\beta$; aequationem invenire. Fig. XXX.

Solutio.

1) Ab angulo D in diagonalem AC demissum perpendiculum Dd, est per solutionem problematis XXVII. (§. 393.) $= d\ fin.\ \beta$, et segmentum $Cd = d\ cof.\ \beta$, segmentum $Ad = \surd(c^2 - d^2 fin.\ \beta^2)$ et hinc tota diagonalis longitudo $= d\ cof.\ \beta + \surd(c^2 + d^2\ fin.\ \beta)$ praeterea

$$fin.\ CAB = \frac{fin.\ \psi \surd(c^2 - d^2 fin.\ \beta^2) - d\ fin.\ \beta\ cof.\ \psi}{c},$$

$$\text{et } cof.\ CAB = \frac{cof.\ \psi \surd(c^2 - d^2 fin.\ \beta^2) + d\ fin.\ \beta\ fin.\ \psi}{c}.$$

2) Ex his et ex finu cofinuque anguli ACB formo finum anguli tertii B eritque

$$fin.\ B = \frac{fin.\ \alpha\ cof.\ \psi \surd(c^2 - d^2 fin.\ \beta^2) + d\ fin.\ \alpha\ fin.\ \beta\ fin.\ \psi}{c}$$

$$+ \frac{cof.\ \alpha\ fin.\ \psi \surd(c^2 - d^2 fin.\ \beta^2) - d\ cof.\ \alpha\ fin.\ \beta\ cof.\ \psi}{c}$$

$$= \frac{fin.\ (\alpha + \psi) \surd(c^2 - d^2 fin.\ \beta^2) - d\ fin.\ \beta\ cof.\ (\alpha + \psi)}{c}.$$

His factis in triangulo finiftro pervenio ad hanc analogiam, $AB : fin.\ ACB = AC : fin.\ B$, h. e. $a : fin.\ \alpha = d\ cof.\ \beta + \surd(c^2 - d^2\ fin.\ \beta^2) :$

$$\frac{fin.\ (\alpha + \varphi) \surd(c^2 - d^2 fin.\ \beta^2) - d\ fin.\ \beta\ cof.\ (\alpha + \psi)}{c},$$

et consequenter ad hanc aequationem pervenitur: $ad\,\text{fin.}\,\beta\,\cos.\,(\alpha+\psi) + cd\,\text{fin.}\,\alpha\,\cos.\,\beta = (a\,\text{fin.}(\alpha+\psi) - c\,\text{fin.}\,\alpha)\sqrt{(c^2-d^2\text{fin.}\,\beta^2)}$. Q. E. I.

Coroll. I.

§. 428. Ex hac aequatione facillime obtinetur: latus $AB = a =$

$$= \frac{c\,\text{fin.}\,\alpha\,(\sqrt{(c^2-d^2\text{fin.}\,\beta^2)} + d\cos.\,\beta)}{\text{fin}\,(\alpha+\psi)\sqrt{(c^2-d^2\,\text{fin.}\,\beta^2)} - d\,\text{fin.}\,\beta\cos.\,(\alpha+\psi)}$$

nec non latus $CD = d =$

$$= \frac{c\,(a\,\text{fin.}\,(\alpha+\psi) - c\,\text{fin.}\,\alpha)}{\sqrt{(a^2\text{fin.}\,\beta^2 \pm 2ac\,\text{fin.}\,\alpha\,\text{fin.}\,\beta\cos.(\alpha+\beta+\psi) + c^2\text{fin.}\,\alpha^2}}.$$

Pro inveniendo latere AD, fit ad abbreviandum $\text{fin.}\,\alpha = h$, $\text{fin.}\,\beta = m$, $\cos.\,\beta = n$, $\text{fin.}\,(\alpha+\psi) = p$; $\cos.\,(\alpha+\psi) = q$, factis substitutionibus proveniet haec aequatio: $admq + cdhn = (ap - ch)\sqrt{c^2-d^2m^2)}$, ex qua operatione absoluta, provenit haec aequatio quarti gradus pro invenienda c, posito ad abbreviandum $mp - nq = -\cos.(\alpha+\beta+\psi) = k$,

$$c^4 - \frac{2apc^3}{h} + \frac{(a^2p^2-d^2)c^2}{h^2} - \frac{2ad^2mkc}{h} - \frac{a^2d^2m^2}{h^2} = 0.$$

Coroll. II.

§. 429. pro inveniendo angulo ACB habetur haec aequatio:

$\text{tang.}\,\alpha =$

$$= \frac{a(\text{fin.}\,\psi\sqrt{(c^2-d^2\,\text{fin.}\,\beta^2)} - d\,\text{fin.}\,\beta\,\cos.\,\psi)}{(c-a\cos.\,\psi)\sqrt{(c^2-d^2\text{fin.}\,\beta^2)} + d(c\cos.\,\beta - a\,\text{fin.}\,\beta\,\text{fin}\,\psi)}.$$

Pro inveniendo angulo A, fit ad abbreviandum $\text{fin.}\,\alpha = h$, $\text{fin.}\,\beta = m$, $\cos.\,\beta = n$, $\text{fin.}\,(\alpha+\psi) = x$, $\cos.(\alpha+\psi) = \sqrt{(1-x^2)}\,\sqrt{(c^2-d^2\,\text{fin.}\,\beta^2} = M$, et factis

factis substitutionibus et transpositione, habetur haec aequatio: $adm\sqrt{(1-x^2)}=Max-Mch-cdhn$, ex qua operatione absoluta, obtinetur:

$$x=\frac{-Mh(M+dn)\pm\sqrt{(a^2d^2m^2-c^2h^2(M+dn)^2)}}{ac}$$

hinc ad angulum ipsum facile descenditur, a quo α subtracto innotescit angulus ψ. Sit ad abbreviandum $\sin.(\alpha+\psi)=m$, $\cos.(\alpha+\psi)=n$, $\sin.\alpha=h$, $am-ch=M$, et $\tang.\beta=t$, pro inveniendo angulo ACD, facta substitutione et operatione absoluta, habetur haec aequatio:

$$t=\frac{-acd^2hn\pm Mc\sqrt{(a^2d^2n^2+(M^2-d^2h^2)(d^2-c^2))}}{a^2d^2n^2+M^2(d^2-c^2)}.$$

Scholion I.

§. 430. Inter sex illos casus qui in aequatione continentur, sunt tres Tetragonometriae proprii, qui obtinent, quando una trium determinationum in triangulo dextro quaeritur; pro latere CD inveniendo solutionem dedi per aequationem quadraticam simplicem; pro angulo ACD aequationem quadraticam affectam tangentialem; lateris AD inventionem ad aequationem quarti gradus perduxi. Singuli casus constituunt in Geometria practica problema novum, cujus pulchritudo cum utilitate certat. Caeteri tres casus utrique methodo et trigonometricae et tetragonometricae subsunt, sed ita ut trigonometrice fere facilius et brevius expediantur, quorum tamen solutiones dedi, quia ex aequatione haud difficulter sequntur.

Coroll. III.

§. 431. Si anguli diagonaliter oppoſiti ſint aequales duobus rectis, ut prodeat trapezium circulo inſcriptibile; aequatio pro latere AB in hanc mutatur:

$$a = \frac{c\, fin.\, \alpha\, (\sqrt{(c^2 - d^2 fin.\, \beta^2)} + d\cos.\, \beta)}{fin.\, \beta\, (\sqrt{(c^2 - d^2 fin.\, \beta^2)} + d\cos.\, \beta)},$$ quae igitur dividendo in hanc abit: $a = \frac{c\, fin.\, \alpha}{fin.\, \beta}$. Verum aequatio pro latere CD, fit $d = \frac{c\,(a\, fin.\, \beta - c\, fin.\, \alpha)}{a\, fin.\, \beta \pm c\, fin.\, \alpha}$, quamobrem, ſiquidem terminus aequationis generalis intermedius denominatoris ſub ſigno radicali ſit affirmativus, ſequitur fore $d = c$, et triangulum ADC aequicrurum. In aequatione quarti gradus pro latere AD, fit $p = m$, et $k = +l$, quae igitur paululum in hanc mutatur:

$$c^4 - \frac{2apc^3}{h} + \frac{(a^2p^2 - d^2)c^2}{h^2} - \frac{2ad^2pc}{h} - \frac{a^2d^2p^2}{h^2} = 0.$$

Coroll. IV.

§. 432. Si angulus A ſit rectus vel etiam aequalis tribus rectis, cadente vertice A ſupra diagonalem BD, et trapezio inverſo prodeunte, pro latere AB aequatio in hanc mutatur:

$$a = \frac{c\, fin.\, \alpha\, (\sqrt{(c^2 - d^2 fin.\, \beta^2)} + d\cos.\, \beta)}{\cos.\, \alpha \sqrt{(c^2 - d^2 fin.\, \beta^2)} \pm d\, fin.\, \beta\, fin.\, \alpha}$$ vel etiam

$$a = \frac{c\, tang.\, \alpha\, (\sqrt{(c^2 - d^2 fin.\, \beta^2)} + d\cos.\, \beta)}{\sqrt{(c^2 - d^2 fin.\, \beta^2)} \pm d\, fin.\, \beta\, tang.\, \alpha};$$ pro latere CD fit:

$$d = \frac{c\,(\pm a\cos.\, \alpha - c\, fin.\, \alpha)}{\sqrt{(a^2 fin.\, \beta^2 \mp 2ac\, fin.\, \alpha\, fin.\, \beta\, (\alpha + \beta) + c^2 fin.\, \alpha^2)}};$$ In

In aequatione quarti gradus pro latere AD fit $p = \pm cos.\, \alpha$, $q = \mp fin.\, \alpha = \mp h$, et $k = \mp fin.(\alpha + \beta)$, quare terminus aequationis fecundus et quartus utrumque fignum accipit, et aequatio fcribitur hoc modo:

$$c^4 \mp \frac{2apc^3}{h} + \frac{(a^2p^2 - d^2)c^2}{h^2} \pm \frac{2ad^2mkc}{h} - \frac{a^2d^2m^2}{h^2} = o.$$

Aequatio pro angulo ACB in hanc mutatur:

$$tang.\, \alpha = \frac{a\sqrt{(c^2 - d^2 fin.\, \beta^2)}}{c\sqrt{(c^2 - d^2 fin.\, \beta^2)} + d(c\, cos.\, \beta \mp a\, fin.\, \beta)}.$$

In aequatione pro angulo ACD fit, $m = \pm cos.\, \alpha$, $n = \mp fin.\, \alpha = \mp h$, et aequatio in hanc paululum immutatur:

$$t = \frac{\pm acd^2h^2 \pm Mc\sqrt{(a^2d^2h^2 + (M^2 - d^2h^2)(d^2 - c^2))}}{a^2d^2h^2 + M^2(d^2 - c^2)}.$$

Coroll. V.

§. 433. Si angulus C fit rectus vel etiam tribus rectis aequalis, cadente vertice C infra diagonalem BD, et trapezio inverfo prodeunte, aequatio pro latere AB in hanc mutatur, α eliminato:

$$a = \frac{c\, cos.\, \beta(\sqrt{(c^2 - d^2 fin.\, \beta^2)} + d)}{cos.(\psi - \beta)\sqrt{(c^2 - d^2 fin.\, \beta^2)} \pm d\, fin.\, \beta\, fin.(\psi - \beta)},$$

fed β eliminato:

$$a = \frac{c\, tang.\, \alpha(\sqrt{(c^2 - d^2 cos.\, \alpha^2)} \mp d)}{fin.(\alpha + \psi)\, fec.\, \alpha\sqrt{(c^2 - d^2 cos.\, \alpha^2)} \mp d\, cos.(\alpha + \psi)}.$$

Verum pro latere CD aequatio in hanc abit:

$$d = \frac{c(a\, fin.(\alpha + \psi) - c\, fin.\, \alpha)}{\sqrt{(a^2 cos.\, \alpha^2 \mp 2ac\, fin.\, \alpha\, cos.\, \alpha\, fin.\, \psi + c^2 fin.\, \alpha^2)}},$$

 fed

ſed α eliminato habetur:

$$d = \frac{\pm c\,(a\cos.(\psi-\beta) - c\cos.\beta)}{\sqrt{(a^2 \sin.\beta^2 \mp 2ac \sin.\beta \cos.\beta \sin.\psi + c^2 \cos.\beta^2)}}.$$

In aequatione quarti gradus fit, $k = \pm \sin.\psi$, $h = \pm n$, et aequatio in hanc mutatur:

$$c^4 \mp \frac{2apc^3}{n} + \frac{(a^2p^2-d^2)c^2}{n^2} - \frac{2ad^2mkc}{n} - \frac{a^2d^2m^2}{n^2} = 0.$$

et in aequatione pro angulo A fit, $h = \pm n$, quae nihil mutatur, niſi quod priori termino dextri membri ſignum utrumque praeponatur.

Coroll. VI.

§. 434. Si angulus A ponatur aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus, et trapezio in triangulum BCD abeunte; aequatio pro AB ſegmento in hanc mutatur:

$$a = \frac{c\,\tang.\alpha\,(\sqrt{(c^2 - d^2 \sin.\beta^2)} + d\cos.\beta)}{\tang.\alpha \sqrt{(c^2 - d^2 \sin.\beta^2)} + d\sin.\beta}.$$

Verum aequatio pro latere CD in hanc tranſit:

$$d = \frac{(a+c)\,c\sin.\alpha}{\sqrt{(a^2 \sin.\beta^2 \mp 2ac \sin.\alpha \sin.\beta \cos.(\alpha+\beta) + c^2 \sin.\alpha^2)}},$$

ſed in aequatione quarti gradus pro latere AD fit, $p = -h$, et $k = -\cos.(\alpha+\beta)$, conſequenter aequatio in hanc paullo ſimpliciorem transformatur:

$$c^4 + 2ac^3 + (a^2-d^2)c^2 + \frac{2ad^2mkc}{h} - \frac{a^2d^2m^2}{h^2} = 0;$$

et aequatio pro angulo ACB in hanc abit:

$\tang.\alpha = ad\sin.\beta : (a+c)\sqrt{(c^2-d^2\sin.\beta^2)} + cd\cos.\beta.$

In aequatione pro angulo ACD fit, $m = -h$, $n = -\cos.\alpha$, $M = -\sin.\alpha\,(a+c) = -h\,(a+c)$,

quae

quae igitur in hanc paululum mutatur:

$$t = \frac{a c d^2 h n \pm M c \sqrt{(a^2 d^2 n^2 + (M^2 - d^2 h^2)(d^2 - c^2))}}{a^2 d^2 n^2 + M^2 (d^2 - c^2)}.$$

Coroll. VII.

§. 435. Si angulus C sit aequalis duobus rectis lateribus CB, CD in diagonalem BD incidentibus et trapezio in triangulum BAD abeunte; aequatio pro latere AB in hanc mutatur α eliminato:

$$a = \frac{c \, \text{sin.} \beta (\sqrt{(c^2 - d^2 \text{sin.} \beta^2)} + d \cos. \beta)}{\text{sin.} (\psi - \beta) \sqrt{(c^2 - d^2 \text{sin.} \beta^2)} + d \, \text{sin.} \beta \cos. (\psi - \beta)},$$

sed β eliminato habetur:

$$a = \frac{c \, \text{sin.} \alpha (\sqrt{(c^2 - d^2 \text{sin.} \alpha^2)} - d \cos. \alpha)}{\text{sin.} (\alpha + \psi) \sqrt{(c^2 - d^2 \text{sin.} \alpha^2)} - d \, \text{sin.} \alpha \cos. (\alpha + \psi)}.$$

Verum aequatio pro latere CD in hanc mutatur:

$$d = \frac{c (a \, \text{sin.} (\alpha + \psi) - c \, \text{sin.} \alpha)}{\text{sin.} \alpha \sqrt{(a^2 + c^2 \pm 2 a c \cos. \psi)}},$$ sed α eliminato:

$$d = \frac{c (\alpha \, \text{sin.} (\psi - \beta) - c \, \text{sin.} \beta)}{\text{sin.} \beta \sqrt{(a^2 + c^2 \pm 2 a c \cos. \psi)}}.$$ In aequatione quarti gradus fit, $h = m$, $k = -\cos. \psi$, quae igitur in hanc paullo simpliciorem mutatur:

$$c^4 - \frac{2 a p c^3}{h} + \left(\frac{a^2 p^2}{h^2} - d^2\right) c^2 + 2 a d^2 k c - a^2 d^2 = 0;$$

denique in aequatione pro angulo A fit etiam $h = m$, quae in hanc mutatur:

$$x = \frac{h}{a c} (-MCM + d n) \pm \sqrt{(a^2 d^2 - c^2 (M + d n)^2)}.$$

Scholion II.

§. 436. Pofitione in penultimo corollario adhibita, folvuntur tria problemata nova, triangula fpectantia, quorum primum obtinet, fi quaeratur angulus ECD, fecundum, fi latus CD, et tertium, fi fegmentum DE; in primo cafu in triangulo totali dantur duo latera et angulus partialis ECB, in fecundo cafu latus et angulus oppofitus, in dertio cafu latus et angulus adjacens et pars BE lateris BD, fed in utroque triangulo partiali dantur duo tantum. Pofitione vero in ultimo corollario adhibita unum ad triangula fpectans novum refolvitur problema, quando quaeritur latus AD, five ad triangulum totale ABD, five partiale AED referatur, cum in hoc non dentur nifi duo, nec in illo nifi duo et fegmentum ED lateris BD; hi quatuor cafus aliis methodis, quam tetragonometricis vix folvuntur. At caeteri cafus, etiam fi nova quadam ratione tetragonometrice folvantur, tamen vulgaribus methodis trigonometricis etiam expediuntur.

Coroll. VIII.

§. 437. Duo dantur cafus in quibus aequatio pro angulo ACD fit rationalis, unus fi fit $c = d$, in quo fit, $tang.\, \beta = \frac{c(-dh \pm (am - ch))}{adn}$, alter fi fit $M = dh$ et confequenter $a = \frac{h(c+d)}{m}$ five $a : c + d = fin.\, \alpha : fin.\, (\alpha + \psi)$, in quo fit: $tang.\, \beta = \frac{acdn(-dh \pm (am - ch))}{a^2 d^2 n^2 + (am - ch)^2 (d^2 - c^2)}$.

Schol.

Scholion III.

§. 438. Si trapezium ita ſit ſive partialiter ſive totaliter inverſum, ut latus AD cadat intra latus AB, valet haec aequatio: $(a\,\text{ſin.}(\alpha+\psi)-c\,\text{ſin.}\,\alpha)\sqrt{}(c^2-d^2\,\text{ſin.}\,\beta^2=c\,d\,\text{ſin.}\,\alpha\,\cos.\beta-a\,d\,\text{ſin.}\,\beta\,\cos.(\alpha-\psi)$. Sin autem ita ſit ſive partialiter ſive totaliter inverſum, ut latus AB cadat intra latus AD, valet haec aequatio: $(a\,\text{ſin.}(\alpha-\psi)-c\,\text{ſin.}\,\alpha)\sqrt{}(c^2-d^2\,\text{ſin.}\,\beta^2)=c\,d\,\text{ſin.}\,\alpha\,\cos\beta-a\,d\,\text{ſin.}\,\beta\,\cos.(\alpha-\psi)$, conſequenter aequatio generalis in hoc caſu pro omni trapezio eſt hujusmodi: $(a\,\text{ſin.}(\alpha\pm\psi)-c\,\text{ſin.}\,\alpha)\sqrt{}(c^2-d^2\,\text{ſin.}\,\beta^2)=c\,d\,\text{ſin.}\,\alpha\,\cos.\beta\pm a\,d\,\text{ſin.}\,\beta\,\cos.(\alpha\pm\psi)$.

Problema XXXI.

§. 439. In figura quadrilatera propoſita $ABCD$ inter haec ſex: latus $AB=a$, $CD=d$, angulum $A=\psi$, $D=\phi$, $ACB=\alpha$, $ACD=\beta$; aequationem invenire Fig. XXXI.

Solutio.

Cum ſit per antecedentia $AC=\dfrac{d\,\text{ſin.}\,\phi}{\text{ſin.}(\beta+\phi)}$ et $\text{ſin.}\,B=-\text{ſin.}(\alpha+\beta+\phi+\psi)$; in triangulo ſiniſtro pervenio ſtatim ad hanc analogiam $AB:\text{ſin.}\,ACB=AC:\text{ſin.}\,B$, h. e. ſymbolis ſubſtitutis

$$a:\text{ſin.}\,\alpha=\frac{d\,\text{ſin.}\,\phi}{\text{ſin.}(\beta+\phi)}:-\text{ſin.}(\alpha+\beta+\phi+\psi)$$ et

conſequenter ad hanc aequationem: $d\,\text{ſin.}\,\alpha\,\text{ſin.}\,\phi=-a\,\text{ſin.}(\beta+\phi)\,\text{ſin.}(\alpha+\beta+\phi+\psi)$. Q. E. I.

Coroll. I.

§. 440. Ex hac aequatione ſtatim in ipſos oculos incurrit eſſe:

latus $AB = a = \frac{-d\,\mathrm{sin.}\,\alpha\,\mathrm{sin.}\,\phi}{\mathrm{sin.}\,(\beta+\phi)\,\mathrm{sin.}\,(\alpha+\beta+\phi+\psi)}$ et

latus $DC = d = \frac{-a\,\mathrm{sin.}\,(\beta+\phi)\,\mathrm{sin.}\,(\alpha+\beta+\phi+\psi)}{\mathrm{sin.}\,\alpha\,\mathrm{sin.}\,\phi}$

ad angulum A obtinendum erit:

$\mathrm{sin.}\,(\alpha+\beta+\phi+\psi) = \frac{-d\,\mathrm{sin.}\,\alpha\,\mathrm{sin.}\,\phi}{a\,\mathrm{sin.}\,(\beta+\psi)}$ et conſequenter $\psi = ang.\,\mathrm{sin.}\frac{-d\,\mathrm{sin.}\,\alpha\,\mathrm{sin.}\,\phi}{a\,\mathrm{sin.}\,(\beta+\phi)} - \alpha - \beta - \phi.$

Verum ſinus a ſe invicem ſeparando alia ſolutio obtinetur; nam facta ſeparatione et poſito ad abbreviandum $\mathrm{sin.}\,\alpha = h$, $\mathrm{sin.}\,\phi = k$, $\mathrm{sin.}\,(\beta+\phi) = m$, $\mathrm{sin.}\,(\alpha+\beta+\phi) = p$, et $\mathrm{cos.}\,(\alpha+\beta+\phi) = q$, $\mathrm{sin.}\,\psi = x$, $\mathrm{cos.}\,\psi = \sqrt{(1-x^2)}$, erit facta transpoſitione haec aequatio abbreviata: $dhk + amqx = -amp\sqrt{(1-x^2)}$ ex qua operatione ad finem perducta obtinetur haec aequatio quadratica affecta:

$$x = \frac{-dhkq \mp p\sqrt{(a^2m^2 - d^2h^2k^2}}{am} = \mathrm{sin.}\,\psi.$$

Coroll. II.

§. 441. Pro angulo ACB obtinendo habetur haec aequatio:

$$tang.\,\alpha = \frac{-a\,\mathrm{sin.}\,(\beta+\phi)\,\mathrm{sin.}\,(\beta+\phi+\psi)}{d\,\mathrm{sin.}\,\phi + a\,\mathrm{sin.}\,(\beta+\phi)\,\mathrm{cos.}\,(\beta+\phi+\psi)}$$

vel etiam haec:

$$tang.\,\alpha = \frac{-a\,tang.\,(\beta+\phi+\psi)}{a + d\,\mathrm{sin.}\,\phi\,\mathrm{sec.}\,(\beta+\phi+\psi)\,\mathrm{cosec.}\,(\beta+\phi)}.$$

Ad

Ad angulum ACD obtinendum aequationem in hanc formam transfundo: $2\,d\,\mathit{fin.}\,\alpha\,\mathit{fin.}\,\phi + a\,\cos.\,(\alpha+\psi) = a\,\cos.\,(\alpha+\psi+2\beta+2\phi)$ unde hanc folutionem confequor:

$$\beta = \tfrac{1}{2}\,ang.\,cos.\frac{2\,d\,\mathit{fin.}\alpha\,\mathit{fin.}\phi + a\cos.(\alpha+\psi)}{a} - \frac{\alpha-2\phi-\psi}{2}.$$

Alia folutio obtinetur finiftro aequationis membro per $\mathit{fin.}\,(\beta+\phi)^2 + \cos.(\beta+\phi)^2 = 1$ multiplicato, ad aequationem refpectu anguli $\beta+\phi$ homogeneam reddendam, et pofito ad abbreviandum $\mathit{fin.}\,\alpha = h$, $\mathit{fin.}\,\phi = k$, $\mathit{fin.}\,(\alpha+\psi) = m$, $\cos.(\alpha+\psi) = n$, et $tang.\,(\beta+\phi) = t$, fic enim facta feparatione, fubftitutione, transpofitione et divifione, prodit $(an+dhk)\,t^2 + amt = -dhk$, ex qua operatione abfoluta obtinetur, haec aequatio:

$$t = \frac{-am \pm \sqrt{(a^2m^2 - 4dhk(an+dhk))}}{2(an+dhk)} = tang.\,(\beta+\phi)$$

hinc ad angulum $\beta+\phi$ facile defcenditur, unde fubtracto angulo dato ϕ relinquitur quaefitus β.

Scholion I.

§. 442. Alia folutio haberetur, fi finiftrum aequationis membrum per $\mathit{fin.}\,\beta^2 + \cos.\,\beta^2$, multiplicaretur, ad aequationem refpectu anguli β homogeneam obtinendam, et $\mathit{fin.}\,\beta$ et $\cos.\,\beta$ in dextro membro a caeteris feparantur, haec autem, quia prolixior evadit, ftudio brevitatis omittitur. Quod fi vero finiftrum aequationis membrum non multiplicetur per $\mathit{fin.}\,(\beta+\phi)^2 + \cos.(\beta+\phi)^2$, maneant vero nomina finuum ad abbreviandum prius data, fitque $\mathit{fin.}\,(\beta+\phi) = x$, et $\cos.(\beta+\phi) = \sqrt{(1-x^2)}$; ad hanc aequationem quadra-

quadraticam, duplici ſigno radicali affectam, et priore complicatiorem pervenio:

$$x = \frac{\sqrt{(am^2 - 2dhkn \pm m\sqrt{(a^2m^2 - 4dhk(dhk + an))})}}{\sqrt{(2a}}.$$

Coroll. III.

§. 443. Pro inveniendo angulo D, ſit ad abbreviandum $\mathit{fin.}\,\alpha = h$, $\mathit{fin.}\,\beta = m$, $\mathit{fin.}\,(\alpha + \beta + \psi) = p$, $\mathit{fin.}\,(\alpha + 2\beta + \psi) = r$, $-\,\mathit{cos.}\,(\alpha + 2\beta + \psi) = s$, $\mathit{fin.}\,\phi = x$, facta ſubſtitutione et omni operatione abſoluta, ad hanc aequationem quarti gradus pervenitur: $x^4 + \frac{2dhs}{a}x^3 + \frac{(d^2h^2 + 2mps - r^2)}{a^2}$

$$x^2 + \frac{2dhmpx + m^2p^2}{a} = 0.$$

Scholion II.

§. 444. Inter ſex caſus qui aequatione continentur; duo ſunt Tetragonometriae proprii, qui obtinent, quando ex angulis D vel ACD inveniendis figura conſtruenda proponitur, et in Geometria practica ſuppeditant duo problemata nova, utilia et ſatis elegantia. Caeteri caſus quatuor, utrique methodo et trigonometricae et tetragonometricae ſubſunt. Verum ſolutiones pro lateribus tetragonometricae, vulgaribus trigonometricis vix cedunt, nec pro angulis A, et ACB, datae trigonometricis multum poſt ponendae.

Coroll. IV.

§. 445. Si anguli A et D ſimul ſumti ſint aequales duobus rectis, ut prodeat trapezium paral-

parallelarum basium AB, DC, aequatio pro latere AB in hanc mutatur:

$$a = \frac{d\ \text{fin.}\ \alpha\ \text{fin.}\ \phi}{\text{fin.}\ (\alpha+\beta)\ \text{fin.}\ (\beta+\phi)},$$ et pro latere

$$DC = d = \frac{a\ \text{fin.}\ (\alpha+\beta)\ \text{fin.}\ (\beta+\phi)}{\text{fin.}\ \alpha\ \text{fin.}\ \phi};$$ aequatio pro angulo ACB in hanc mutatur:

$$\text{tang.}\ \alpha = \frac{a\ \text{fin.}\ \beta\ \text{fin.}\ (\beta+\phi)}{d\ \text{fin.}\ \phi - a\ \text{cos.}\ \beta\ \text{fin.}\ (\beta+\phi)}$$

$$= \frac{a\ \text{tang.}\ \beta}{d\ \text{fin.}\ \phi\ \text{fec.}\ \beta\ \text{cos.}\ \text{fec.}\ (\beta+\phi) - a}.$$ Pro angulo ACD aequatio prior in hanc mutatur:

$$\beta = \tfrac{1}{2}\ \text{ang. cos.} \frac{-2\lambda\ \text{fin.}\ \alpha\ \text{fin.}\ \phi - a\ \text{cos.}\ (\alpha+\phi)}{a} - \frac{\alpha-\phi}{2}$$

at aequationis posterioris forma externa invariata manet. Si anguli D et C, simul sumti duobus rectis aequales ponantur, ut prodeat trapezium parallelarum basium AD, BC; aequatio pro latere AB in hanc abit: $\text{fin.}\ \alpha\ \text{fin.}\ \phi$, $\psi = a = \frac{d\ \text{fin.}\ \phi}{\text{fin.}\ \psi}$ et pro latere $DC = d = \frac{a\ \text{fin.}\ \psi}{\text{fin.}\ \phi}$, et consequenter pro angulo A fit, $\psi = \text{ang. fin.} \frac{d\ \text{fin.}\ \phi}{a}$; ex aequatione quadratica pro angulo A data idem elicitur. Quod si anguli diagonaliter oppositi A et C ponantur simul sumti aequales duobus rectis, ut prodeat trapezium circulo inscriptibile, habetur pro latere $AB = a = \frac{d\ \text{fin.}\ \alpha}{\text{fin.}\ (\beta+\phi)}$, et latere DC

$DC = d = \frac{a\,fin.\,(\beta+\varphi)}{fin.\,\alpha}$ et hinc elicitur pro obtinende angulo D, $\varphi = ang.\,fin.\frac{d\,fin.\,\alpha}{a} - \beta$.

Coroll. V.

§. 446. Si angulus A fit rectus vel etiam tribus rectis aequalis, cadente vertice A fupra diagonalem BD et trapezio inverfo prodeunte; habetur pro latere AB, $a = \frac{\mp d\,fin.\,\alpha\,fin.\,\varphi}{fin.\,(\beta+\psi)\,cos.\,(\alpha+\beta+\varphi)}$

et pro latere CD, $d = \frac{\mp a\,fin.\,(\beta+\varphi)\,cos.\,(\alpha+\beta+\varphi)}{fin.\,\alpha\,fin.\,\varphi}$,

pro angulo $A\,c\,B$ habetur:

$tang.\,\alpha = \frac{\mp a\,fin.\,(\beta+\varphi)\,cos.\,(\beta+\varphi)}{d\,fin.\,\varphi \mp a\,fin.\,(\beta+\varphi)^2}$ vel etiam

$tang.\,\alpha = \frac{\mp a\,cos.\,(\beta+\varphi)}{d\,fin.\,\varphi\,cofec.\,(\beta+\varphi) \mp a\,fin.\,(\beta+\varphi)}$,

pro angulo ACD, fit ex priori aequatione:

$\beta = \frac{1}{2}\,ang.\,fin.\frac{\mp 2d\,fin.\,\alpha\,fin.\,\varphi + a\,fin.\,\alpha}{a} - \frac{\alpha - 2\varphi}{2}$,

fed in altera aequatione quadratica fit $m = \pm cos.\,\alpha$, $n = \mp fin.\,\alpha = \mp h$, et aequatio in hanc paululum mutatur: $t = \frac{\mp am \pm \sqrt{(a^2m^2 - 4dh^2k\,(dk \mp a))}}{2h\,(dk \mp a)}$.

Denique in aequatione pro angulo D, fit, $p = \pm\,cos.\,(\alpha+\beta)$, $r = \pm\,cos.\,(\alpha + 2\beta)$, $s = \mp\,fin.\,(\alpha + 2\beta)$, quare aequatio quoad fignum termini fecundi mutatur hoc modo:

$$x^4 - \frac{2dhs}{a}x^3 + \left(\frac{d^2h^2}{a^2} + 2mps - r^2\right)x^2 + \frac{2dhmp}{a}x + mp^2 = 0.$$

Coroll.

Coroll. VI.

§. 447. Si ponatur angulus D rectus vel etiam tribus rectis aequalis, cadente vertice D ad finistram diagonalis et trapezio inverso prodeunte, expressio pro latere AB in hanc abit:

$a = \frac{\mp d \text{ fin. } \alpha}{\cos. \beta \cos. (\alpha+\beta+\psi)}$, et pro latere DC

$d = \frac{\mp a \cos. \beta \ \cos. (\alpha+\beta+\psi)}{\text{fin. } \alpha}$, pro angulo A aequatio prior in hanc mutatur:

$\psi = ang. \cos. \frac{\mp d \text{ fin. } \alpha}{a \cos. \beta} - \alpha - \beta$, fed in posteriore quadratica fit, $k = \pm 1$, $m = \pm \cos. \beta$, $p = \pm \cos. (\alpha+\beta)$, $q = \mp \text{fin.} (\alpha+\beta)$, quae igitur in hanc paululum immutatur:

$x = \frac{\pm dhq \mp p \sqrt{(a^2 m^2 - d^2 h^2)}}{am}$. Verum aequatio pro angulo ACB in hanc abit:

$$tang. \alpha = \frac{a \cos. \beta \cos. (\beta+\psi)}{a \cos. \beta \text{ fin. } (\beta+\psi) \mp d}$$

$$= \frac{a}{a \ tang. (\beta+\psi) \mp d \text{ fec. } \beta \text{ fec. } (\beta+\psi)};$$

aequatio pro angulo ACD in hanc transformatur:

$$\beta = \tfrac{1}{2} ang. fin. \frac{-2d \text{ fin. } \alpha \mp a \cos. (\alpha+\psi)}{a} - \frac{\alpha - \psi}{2}$$

fed in altera aequatione quadratica fit, $k = \pm 1$, quae in hanc paululum immutatur:

$$tot. \beta = \frac{am \mp \sqrt{(a^2 m^2 - 4dh (dh \pm an))}}{2(an \pm dh)}.$$

Coroll.

Coroll. VII.

§. 448. Quod si angulus C sit rectus vel etiam aequalis tribus rectis, cadente C, infra diagonalem BD, et trapezio inverso prodeunte, aequatio pro latere AB, in hanc mutatur:

$$a = \frac{\mp d \,\text{fin.}\, \alpha \,\text{fin.}\, \phi}{\text{fin.}(\beta+\phi)\cos.(\phi+\psi)} = \frac{\mp d \cos.\beta \,\text{fin.}\, \phi}{\text{fin.}(\beta+\phi)\cos.(\phi+\psi)}$$

$$= \frac{\mp d \,\text{fin.}\, \alpha \,\text{fin.}\, \phi}{\cos.(\phi-\alpha)\cos.(\phi+\psi)},$$ in uno casu α, altero β eliminato, consequenter CD fit in primo casu

$$d = \frac{\mp a \,\text{fin.}(\beta+\phi)\cos.(\phi+\psi)}{\text{fin.}\, \alpha \,\text{fin.}\, \phi},$$ in secundo casu,

$$d = \frac{\mp a \,\text{fin.}(\beta+\phi)\cos.(\phi+\psi)}{\cos.\beta \,\text{fin.}\, \phi},$$ et in tertio,

$$d = \frac{\mp a \cos.(\phi-\alpha)\cos.(\phi+\psi)}{\text{fin.}\, \alpha \,\text{fin.}\, \phi},$$ hinc sequitur fore pro angulo A ex aequatione priore:

$$\psi = \text{ang. cos.} \frac{\mp d \,\text{fin.}\, \alpha \,\text{fin.}\, \phi}{a \,\text{fin.}(\beta+\phi)} - \phi$$ vel etiam

$$\psi = \text{ang. cos.} \frac{\mp d \cos.\beta \,\text{fin.}\, \phi}{a \,\text{fin.}(\beta+\phi)} - \phi$$

$$= \text{ang. cos.} \frac{\mp d \,\text{fin.}\, \alpha \,\text{fin.}\, \phi}{a \cos.(\phi-\alpha)} - \phi.$$ Sed in aequatione quadratica fit, $p = \pm \cos.\phi$, $q = \mp \text{fin.}\, \phi = \mp k$, quae igitur in hanc paululum mutatur: $x = \frac{\pm d h k^2 \mp p \sqrt{(a^2 m^2 - d^2 h^2 k^2)}}{a m}$.

Sed in aequatione quarti gradus pro angulo D, fit $p = \pm \cos.\psi$, $r = \pm \cos(\beta+\psi)$, $f = \pm \text{fin.}(\beta+\phi)$, quae igitur quoad signa paululum mutatur in sequen-

sequentem $x^4 \pm \frac{2dhsx^3}{a} + \frac{(d^2h^2 - 2mps - r^2)}{a^2}$

$\pm \frac{2dhmpx}{a} + m^2p^2 = 0.$

Coroll. VIII.

§. 449. Si angulus A ponatur aequalis duobus rectis, lateribus AD, AB in diagonalem BD incidentibus, et trapezio in triangulum BCD abeunte, habetur pro lateris BD segmento EB haec aequatio:

$a = \frac{d \, fin. \, \alpha \, fin. \, \phi}{fin. \, (\beta + \phi) \, fin. \, (\alpha + \beta + \phi)}$ et hinc pro latere CD, $d = \frac{a \, fin. \, (\beta + \phi) \, fin. \, (\alpha + \beta + \phi)}{fin. \, \alpha \, fin. \, \phi}$. Pro angulo ACB habetur haec:

$tang. \, \alpha = \frac{a \, fin. \, (\beta + \phi)^2}{d \, fin. \, \phi - a \, fin. \, (\beta + \phi) \, cos. \, (\beta + \phi)}$ vel etiam

$tang. \, \alpha = \frac{a}{d \, fin. \, \phi \, cosec. \, (\beta + \phi)^2 - a \, cot. \, (\beta + \phi)}$.

aequatio prior pro angulo ACD in hanc mutatur:

$\beta = \frac{1}{2} \, ang. \, cos. \frac{-2 d \, fin. \, \alpha \, fin. \, \phi + a \, cos. \, \alpha}{a} - \frac{\alpha - 2 \phi}{2}$

et cum sit $m = - fin. \, \alpha = - h$, et $n = - cos. \, \alpha$, posterior quadratica fit:

$t = \frac{ah \pm \sqrt{(a^2h^2 - 4dhk(dhk - an))}}{2(dhk - an)}$. In aequatione quarti gradus pro angulo D fit, $p = - fin. (\alpha + \beta)$, $r = - fin. \, (\alpha + 2\beta)$, $s = + cos. (\alpha + 2\beta)$, quae igitur quoad signa paululum in hanc mutatur:

$x^4 - \frac{2dhsx^3}{a} + \frac{(d^2h^2 - 2mps - r^2)x^2}{a^2} - \frac{2dhmpx}{a} + m^2p^2 = 0,$

U Coroll.

Coroll. IX.

§. 450. Si angulus C sit aequalis duobus rectis, lateribus CB, CD in diagonalem BD incidentibus et trapezio in triangulum BAD abeunte; aequatio pro latere AB in hanc mutatur:

$$a = \frac{d\,\sin.\alpha\,\sin.\phi}{\sin.(\beta+\phi)\,\sin.(\phi+\psi)} = \frac{d\,\sin.\beta\,\sin\,\phi}{\sin.(\beta+\phi)\,\sin.(\phi+\psi)}$$

$$= \frac{d\,\sin\,\alpha\,\sin.\phi}{\sin.(\phi-\alpha)\,\sin.(\phi+\psi)},$$ angulis α et β sigillatim eliminatis, hinc pro lateris BD segmento ED habetur: $d = \frac{a\,\sin.(\beta+\phi)\sin.(\phi+\psi)}{\sin.\alpha\,\sin.\phi.}$ vel etiam

$$d = \frac{a\,\sin\,(\beta+\phi)\sin.(\beta+\phi)}{\sin.\beta\,\sin.\phi} = \frac{a\,\sin.(\phi-\alpha)\sin.(\phi+\psi)}{\sin.\alpha\,\sin.\phi},$$

igitur pro angulo A fit:

$$\psi = ang.\sin.\frac{d\,\sin.\alpha\,\sin.\phi}{a\,\sin.(\sin.\beta+\phi)} - \phi$$ vel etiam

$$\psi = ang.\ \sin.\frac{d\,\sin.\ \beta\ \sin.\ \phi - \phi}{a\,\sin.(\beta+\phi)}$$

$$= ang.\sin.\frac{d\,\sin.\alpha\ \sin.\phi}{a\ \sin.\ (\phi-\alpha)} - \phi.$$ Sed in aequatione quadratica fit, $p = -k$, $q = -\cos.\phi$, quae igitur in hanc perparum mutatur:

$$x = \frac{k}{am}(dhq \mp \surd(a^2m^2 - d^2h^2k^2)),$$ vel etiam

$$x = \frac{dhkq \mp k\surd(a^2m^2 - d^2h^2k^2)}{am}.$$ Denique in aequatione quarti gradus pro angulo D fit, $p = -\sin.\psi$, $f = +\cos.(\beta+\psi)$, $r = -\sin.(\beta+\psi)$, forma vero aequationis erit eadem cum illa corollarii praecedentis.

Schol.

Scholion III.

§. 451. Per poſitionem in octavo corollario adhibitam, duo ad triangula ſpectantia problemata nova ſolvuntur, quae obtinent quando anguli D et ECD quaeruntur; cum in neutro caſu dentur, nec in triangulis partialibus nec in totali niſi duo cum ſegmento BE lateris BD et parte ECB anguli C, caeteri tres caſus etiam nova quadam ratione ſolvuntur, licet methodis vulgaribus trigonometricis etiam expediantur.

Per poſitionem in coroll. IX. adhibitam, unicus obtinetur caſus ad triangula ſpectans, qui non niſi Tetragonometrice ſolvitur, qui tum obtinet, quando angulus D quaeritur, ſive ad triangulum totale ABD, ſive ad partiale AED referatur, quia neque in partialibus, neque in totali dantur niſi duo cum ſegmento ED lateris BD; caeteri caſus licet Tetragonometrice ſolvantur, tamen etiam Trigonometrice expediuntur.

Coroll. X.

§. 452. Si anguli diagonaliter oppoſiti A et C, ſint duobus rectis aequales, ut trapezium ſit circulo inſcriptibile, in aequatione quarti gradus ſit $p=o$, $r=fin.\beta=m$, et $s=-cos.\beta$, conſequenter aequatio ad gradum ſecundum deprimitur, fitque $a^2x^2+2adhsx+d^2h^2-a^2r^2=o$, ex qua abſoluta operatione obtinetur haec aequatio valoribus reſtitutis:

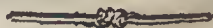
$$fin.\varphi = \frac{-d\,fin.\alpha\,cos.\beta \pm fin.\beta\,\sqrt{(a^2-d^2fin.\alpha^2)}}{a},$$

 quae

quae etiam ex aequatione $d = \frac{a\, \text{fin}\, (\beta + \varphi)}{\text{fin.}\ \alpha}$ ftatim elici potuiffet in fine coroll. IV.

Scholion IV.

§. 453. Quod fi latus AD cadat intra latus AB, five trapezium fit partialiter five totaliter inverfum, valet haec aequatio: $d\, \text{fin.}\, \alpha\, \text{fin.}\, \varphi = a\, \text{fin.}\, (\beta + \varphi)\, \text{fin.}\, (\beta + \varphi - \psi - \alpha)$. Sin vero latus AB cadat intra latus AD five trapezium fit partialiter five totaliter inverfum, valet haec aequatio: $\text{fin.}\, \alpha\, \text{fin.}\, \varphi = a\, \text{fin.}\, (\beta + \varphi)\, \text{fin.}\, (\beta + \varphi - \psi - \alpha)$, quare aequatio generaliffima pro omni trapezio in hoc cafu eft hujusmodi: $d\, \text{fin.}\, \alpha\, \text{fin.}\, \varphi = \mp a\, \text{fin.}\, (\beta + \varphi)\, \text{fin.}\, (\beta + \varphi \pm \psi \pm \alpha)$.



CAPUT

CAPUT IX.

Continens tria problemata tertiae classis particularis, sub posteriore principali contentae.

Problema XXXII.

§. 454. In proposita figura quadrilatera $ABCD$, inter haec sex: latus $BC=b$, $AD=c$, angulos $A=\psi$, $B=\lambda$, $D=\varphi$, $ACD=\beta$, aequationem invenire Fig. XXXII.

Solutio.

Cum sit ex (§. 377.) $AC=\frac{c\,\text{sin.}\,\varphi}{\text{sin.}\,\beta}$, et $\text{sin.}\,CAB=-\,\text{sin.}\,(\beta+\varphi+\psi)$; in triangulo sinistro pervenio statim ad hanc analogiam BC: $\text{sin.}\,CAB=AC$: $\text{sin.}\,B$, h. e. symbolis substitutis $b:-\,\text{sin.}\,(\beta+\varphi+\psi)=\frac{c\,\text{sin.}\,\varphi}{\text{sin.}\,\beta}:\text{sin.}\,\lambda$, hinc emergit haec aequatio: $b\,\text{sin.}\,\beta\,\text{sin.}\,\lambda=-\,c\,\text{sin.}\,\varphi\,\text{sin.}\,(\beta+\varphi+\psi)$, Q. E. I.

Coroll. I.

§. 455. Ex hac aequatione statim in ipsos oculos incurrit esse:

latus $BC=b=\frac{-\,c\,\text{sin.}\,\varphi\,\text{sin.}\,(\beta+\varphi+\psi)}{\text{sin.}\,\beta\,\text{sin.}\,\lambda}$, et

latus $AD=c=\frac{-\,b\,\text{sin.}\,\beta\,\text{sin.}\,\lambda}{\text{sin.}\,\varphi\,\text{sin.}\,(\beta+\varphi+\psi)}$,

 et

et pro angulo ACD ſtatim habetur:
$$\textit{fin.}\, \lambda = \frac{-c\, \textit{fin.}\, \phi\, \textit{fin.}\, (\beta+\phi+\psi)}{b\, \textit{fin.}\, \beta},$$ et hinc angulus ipſe $\lambda = \textit{ang. fin.} \frac{-c\, \textit{fin.}\, \phi\, \textit{fin.}\, (\beta+\phi+\psi)}{b\, \textit{fin.}\, \beta}$, conſimiliter obtinetur angulus A, cum ſit
$$\textit{fin.}\, (\beta+\phi+\psi) = \frac{-b\, \textit{fin.}\, \beta\, \textit{fin.}\, \lambda}{c\, \textit{fin.}\, \phi}$$ et conſequenter
$$\psi = \textit{ang. fin.} \frac{-b\, \textit{fin.}\, \beta\, \textit{fin.}\, \lambda}{c\, \textit{fin.}\, \phi} - \beta - \phi.$$

Coroll. II.

§. 456. Pro obtinendo angulo ACD ad hanc aequationem pervenio:
$$\textit{tang.}\, \beta = \frac{-c\, \textit{fin.}\, \phi\, \textit{fin.}\, (\phi+\psi)}{b\, \textit{fin.}\, \lambda + c\, \textit{fin.}\, \phi\, \textit{cos.}\, (\phi+\psi)}$$ vel etiam
$$\textit{tang.}\, \beta = \frac{-c\, \textit{fin.}\, \phi\, \textit{tang.}\, (\phi+\psi)}{c\, \textit{fin.}\, \phi + b\, \textit{fin.}\, \lambda\, \textit{fec.}\, (\phi+\psi)}$$
$$= \frac{-c\, \textit{tang.}\, (\phi+\psi)}{c + b\, \textit{fin}\, \lambda\, \textit{cofec.}\, \phi\, \textit{fec.}\, (\phi+\psi)}.$$ Ad obtinendum angulum D aequationem in hanc formam transfundo: $2b\, \textit{fin.}\, \lambda\, \textit{fin.}\, \beta + c\, \textit{cos.}\, (\beta+\psi) = c\, \textit{cos.}\, (\beta+\psi+2\phi)$, hinc ad ipſos angulos deſcendendo obtineo hanc aequationem:
$$\phi = \tfrac{1}{2}\, \textit{ang. cos.} \frac{2b\, \textit{fin.}\, \lambda\, \textit{fin.}\, \beta + c\, \textit{cos.}\, (\beta+\psi)}{c} - \frac{\beta-\psi}{2}.$$

Coroll. III.

§. 457. Altera ſolutio obtinetur multiplicando ſiniſtrum aequationis membrum per $\textit{fin.}\, \phi^2 + \textit{cos.}\, \phi^2$ ad aequationem reſpectu anguli

guli ϕ homogeneam reddendam, et brevitatis gratia ponendo: $fin.\lambda=h$, $fin.\beta=k$, $fin.(\beta+\psi)=m$, $cos.(\beta+\psi)=n$, quare facta separatione, transpositione, substitutione et divisione habetur: $(bhk+cn)x^2+cmx=-bhk$, ex qua operatione absoluta obtinetur haec aequatio:

$$tang.\,\phi=\frac{-cm\pm\sqrt{(c^2m^2-4bhk(cn+bhk))}}{2(cn+bhk)}.$$

Scholion I.

§. 458. Sine hoc artificio alia solutio licet operosior obtinetur. Sinus tantum a se invicem separando, et positionibus ut prius factis, sit $fin.\,\phi=x$ et $cos.\,\phi=\sqrt{(1-x^2)}$, factaque substitutione et transpositione habetur haec aequatio abbreviata: $bhk+cnx^2=-cmx\sqrt{(1-x^2)}$. ex qua operatione ad finem perducta, obtinetur haec aequatio quadratica:

$$x=\frac{\sqrt{(cm^2-2bhkn\pm m\sqrt{(c^2m^2-4bhk(bhk+cn))})}}{\sqrt{c}}.$$

Scholion II.

§. 459. Inter sex casus qui in aequatione continentur, duo sunt Tetragonometriae proprii, qui tunc obtinent, quando figura ex inveniendis angulis in triangulo dextro construenda proponitur, qui casus in Geometria practica duo problemata nova satis pulchra et utilia suppeditant. Caeteri quatuor casus utrique methodo et tetragonometricae et trigonometricae subsunt, quorum tamen solutiones tetragonometricae, trigonometricis parum aut nihil cedunt.

Coroll. IV.

§. 460. Si anguli A et D ſint ſimul ſumti aequales duobus rectis, prodeunte trapezio parallelarum baſium AB, CD, expreſſio pro latere BC in hanc mutatur: $b = \frac{c\, \text{ſin.}\, \phi}{\text{ſin.}\, \lambda}$; ſi vero anguli A et B ſimul ſumti ſint aequales duobus rectis, prodeunte trapezio parallelarum baſium AD, BC; aequatio pro latere BC, eliminato λ, in hanc mutatur: $b = \frac{-c\, \text{ſin.}\, \phi\, \text{ſin.}\, (\beta + \phi + \psi)}{\text{ſin.}\, \beta\, \text{ſin.}\, \psi}$, ψ vero eliminato habetur:
$b = \frac{c\, \text{ſin.}\, \phi\, \text{ſin.}\, (\beta + \phi - \lambda)}{\text{ſin.}\, \beta\, \text{ſin.}\, \lambda}$, hinc erit
$c = \frac{-b\, \text{ſin.}\, \beta\, \text{ſin.}\, \psi}{\text{ſin.}\, \phi\, \text{ſin.}\, (\beta + \phi + \psi)}$, vel etiam
$c = \frac{b\, \text{ſin.}\, \beta\, \text{ſin.}\, \lambda}{c\, \text{ſin.}\, \phi\, \text{ſin.}\, (\beta + \phi - \lambda)}$. Pro angulo ACD habetur, eliminato λ, haec expreſſio:
$\text{tang.}\, \beta = \frac{-c\, \text{ſin.}\, \phi\, \text{ſin.}\, (\phi + \psi)}{b\, \text{ſin.}\, \psi + c\, \text{ſin.}\, \phi \cos.\, (\phi + \psi)}$, vel etiam
$\text{tang.}\, \beta = \frac{-c\, \text{tang.}\, (\phi + \psi)}{c + b\, \text{ſin.}\, \psi\, \text{coſec.}\, \phi\, \text{ſec.}\, (\phi + \psi)}$, aequatio angularis pro angulo D in hanc mutatur, λ eliminato:

$$\phi = \tfrac{1}{2}\, \text{ang.}\, \cos.\, \frac{2b\, \text{ſin.}\, \psi\, \text{ſin.}\, \beta + \cos.\, (\beta + \psi)}{c} - \frac{\beta - \psi}{2}$$

vel etiam, ſi eliminetur ψ, prodit haec altera:

$$\phi = \tfrac{1}{2}\, \text{ang.}\, \cos.\, \frac{-2b\, \text{ſin.}\, \lambda\, \text{ſin.}\, \beta + c \cos.\, (\beta - \lambda)}{c} - \frac{\beta + \lambda}{2}.$$

Si

Si vero anguli diagonaliter oppositi B et D sint aequales duobus rectis, ut prodeat trapezium circulo inscriptibile, pro latere BC prodit haec aequatio: $b = \frac{-c\,fin.\,(\beta+\phi+\psi)}{fin.\,\beta} = \frac{c\;fin.\,(\beta+\psi-\lambda)}{fin.\,\beta}$ angulo ϕ eliminato, et hinc habetur: latus $AD = c = \frac{-b\;fin.\,\beta}{fin.\,(\beta+\phi+\psi)} = \frac{b\;fin.\,\beta}{fin.\,(\beta+\psi-\lambda)}$. Pro angulo ACD habetur haec expressio, angulo λ eliminato: $tang.\,\beta = \frac{-\,c\;fin.\,(\phi+\psi)}{b+c\;cos.\,(\phi+\psi)}$ vel etiam $tang.\,\beta = \frac{-\,c\;tang.\,(\phi+\psi)}{c+b\;fec.\,(\phi+\psi)}$. Angulo autem ϕ eliminato habetur:
$tang.\,\beta = \frac{c\,fin.\,(\psi-\lambda)}{b-c\;cos.\,(\psi-\lambda)} = \frac{c\;tang.\,(\psi-\lambda)}{b\,fec.\,(\psi-\lambda)-c}$, denique pro angulo A habetur:
$\psi = ang.fin.\frac{-b\,fin.\,\beta}{c} = ang.\,fin.\frac{b\,fin.\,\beta}{c} - \beta - \lambda$.

Coroll. V.

§. 461. Si angulus A sit rectus vel etiam aequalis tribus rectis, ut A cadat supra diagonalem BD, et prodeat trapezium inversum, aequatio pro latere BC in hanc mutatur: $b = \frac{\mp\, c\;fin.\,\phi\;cos.\,(\beta+\phi)}{fin.\,\beta\,fin.\,\lambda}$ et hinc pro latere $AD = c = \frac{\mp\, b\,fin.\,\beta\,fin.\,\lambda}{c\,fin.\,\phi\;cos.\,(\beta+\phi)}$ et pro angulo B habetur: $fin.\,\lambda = \frac{\mp\, c\cdot fin.\,\phi\;cos.\,(\beta+\phi)}{b\;fin.\,\beta}$, tum

 etiam

etiam pro angulo ACD habetur haec expressio:

$$tang.\beta = \frac{\mp c\ fin.\psi\, cos.\phi}{b\ fin.\lambda - c\ fin.\phi^2} = \frac{\mp c}{b\ fin.\lambda\ fec\ \phi\, cofec.\phi - c\ tang.\phi}.$$

Denique pro angulo D habetur haec aequatio angularis:

$$\psi = \tfrac{1}{2}\ ang.\ fin.\frac{\mp 2\, b\ fin.\ \lambda\ fin.\ \beta + c\ fin.\ \beta}{c} - \frac{\beta}{2}$$

Sed in aequatione quadratica coroll. III. fit: $m = \pm\ cos.\ \beta$, $n = \mp\ fin.\ \beta = \mp\, k$, quae igitur in hanc paululum immutatur:

$$tang.\phi = \frac{\mp\, c\, m}{} == \frac{\sqrt{(c^2 m^2 - 4bhk^2(bh\mp c))}}{2k(bh\mp c)}.$$

Coroll. VI.

§. 462. Si angulus D fit rectus vel etiam tribus rectis aequalis, cadente vertice D ad finiftram diagonalis AC, et trapezio inverfo prodeunte, habetur pro latere BC haec aequatio:

$b == \dfrac{-c\ cos\ (\beta + \psi)}{fin.\ \beta\ fin.\ \lambda}$, confequenter pro latere

$AD = c = \dfrac{-b\ fin.\ \beta\ fin.\ \lambda}{cos.\ \beta + \psi)}$, nec non pro angulo A

$\psi = ang.\ cos.\dfrac{\mp b\ fin.\ \beta\ fin.\ \lambda}{c} - \beta$, pro angulo ACD

habetur: $tang.\beta = \dfrac{-\ c\ fin.\ \psi}{b\ fin.\lambda - fin.\psi} == \dfrac{-\ c}{b\ fin.\lambda\ cofec.\psi - c}$;

denique pro angulo B habetur haec expressio:

$$fin.\ \lambda = \frac{-\ c\ cos\ \ \beta + \psi)}{b\ fin.\ \beta.}.$$

Coroll.

Coroll. VII.

§. 463. Si angulus B sit rectus vel etiam tribus rectis aequalis, cadente vertice B ad dextram diagonalis AC, et trapezio inverso prodeunte, habetur pro latere BC

$b = \frac{\mp c \sin. \varphi \sin. (\beta + \varphi + \psi)}{\sin. \beta}$ et pro latere AD

$c = \frac{\mp b \sin. \beta}{\sin. \varphi \sin. (\beta + \varphi + \psi)}$, pro angulo A,

$\psi = ang. \sin. \frac{\mp b \sin. \beta}{c \sin. \varphi} - \beta - \varphi$, pro angulo ACD

habetur: $tang. \beta = \frac{-c \sin. \varphi \sin. (\varphi + \psi)}{c \sin. \varphi \cos. (\varphi + \psi) \mp b}$ vel etiam

$tang. \beta = \frac{-c\, tang. (\varphi + \psi)}{c \pm b\, cosec. \varphi\, sec. (\varphi + \psi)}$, et pro angulo D

$\varphi = \frac{1}{2} ang. \cos. \frac{\pm 2 b \sin. \beta + c \cos. (\beta + \psi)}{c} - \frac{\beta - \psi}{2}$.

Sed in aequatione quadratica fit, $h = \pm 1$, quae igitur in hanc paululum mutatur:

$tang. \varphi = \frac{-c m \pm \sqrt{(c^2 m^2 - 4 b k (b k \pm c n))}}{2 (c n \pm b k)}$.

Coroll. VIII.

§. 464. Si angulus A ponatur aequalis duobus rectis, lateribus AB, AD in diagonalem BD cadentibus et trapezio in triangulum BCD abeunte fit: latus $BC = b = \frac{c \sin. \varphi \sin. (\beta + \varphi)}{\sin. \beta \sin. \lambda}$, et

segmentum DE lateris BD, $c = \frac{b \sin. \beta \sin. \lambda}{\sin. \varphi \sin. (\beta + \varphi)}$,

pro

pro angulo B habetur: $fin.\lambda = \frac{c\,fin.\varphi\,fin.(\beta+\varphi)}{b\,fin.\beta}$ et pro angulo ECD, $tang\,\beta = \frac{c\,fin.\varphi^2}{b\,fin.\lambda - c\,fin.\varphi\,cos.\varphi}$ $= \frac{c\,tang.\varphi}{b\,fin.\lambda\,fec.\varphi\,cofec.\varphi - c}$. Pro angulo D fit aequatio angularis:

$$\varphi = \tfrac{1}{2}\,ang.\,cos.\frac{-2\,b\,fin.\lambda\,fin.\beta + c\,cos.\beta}{c} - \frac{\beta}{2},$$

fed in aequatione quadratica fit, $m = -fin.\beta = -k$, $n = -cos.\beta$, quae igitur in hanc paululum mutatur:

$$tang.\varphi = \frac{ck \pm \sqrt{(c^2k^2 - 4bhk(bhk - cn))}}{2(bhk - cn)}.$$

Scholion III.

§. 465. Pofitione in hoc corollario adhibita, folvitur unum ad triangula fpectans problema, quod Trigonometrice vel aliis methodis vix folvitur, et tunc obtinet quando angulus D quaeritur, nam in triangulo totali non dantur nifi duo, fcilicet latus BC, et angulus B, nec in partialibus triangulis dantur nifi duo. Caeteri cafus licet Tetragonometrice folvantur, tamen etiam trigonometrice expediuntur.

Scholion IV.

§. 466. Si latus AD cadat intra latus AB, five trapezium fit partialiter five totaliter inverfum, erit aequatio: $b\,fin.\,\beta\,fin.\,\lambda = c\,fin.\,\varphi\,fin.(\beta+\varphi-\psi)$. Si vero latus AB cadat intra latus AD five trapezium fit partialiter five totaliter

liter inverſum, erit aequatio: $b \, fin. \lambda \, fin. \beta = c \, fin. \phi \, fin. (\beta + \phi + \psi)$, conſequenter eſt aequatio generalis: $b \, fin. \lambda \, fin. \beta = \mp c \, fin. \phi \, fin. (\beta + \phi \pm \psi)$.

Problema XXXIII.

§. 467. In figura quadrilatera propoſita $ABCD$, inter haec ſex: latus $AB = a$, $AD = c$, $BC = b$, angulum $A = \psi$, $ACD = \beta$, $D = \phi$; aequationem invenire Fig. XXXIII.

Solutio.

Cum ſit per antecedentia $AC = \frac{c \, fin. \phi}{fin. \beta}$ et $cos. \; CAB = - cos. (\beta + \phi + \psi)$, erit ab angulo B in AC perpendiculo demiſſo ſegmentum $Ab = - a \, cos. (\beta + \phi + \psi)$, et conſequenter per (Eucl. II. 11.) ad hanc aequationem pervenio: $b^2 = \frac{a^2 + c^2 \, fin. \phi^2}{fin. \beta^2} + \frac{2ac \, fin. \phi \, cos. (\beta + \phi + \psi)}{fin. \beta}$ adeoque habetur: $b^2 \, fin. \beta^2 = a^2 \, fin. \beta^2 + c^2 \, fin. \phi^2 \pm 2ac \, fin. \beta \, fin. \phi \, cos. (\beta + \phi + \psi)$, Q. E. I.

Coroll. I.

§. 468. Ex hac aequatione ſtatim obtinetur latus BC, cum ſit: $b = \frac{\sqrt{(a^2 \, fin. \beta^2 \pm 2ac \, fin. \beta \, fin. \phi \, cos. (\beta + \phi + \psi) + c^2 \, fin. \psi^2)}}{fin. \beta}$ vel etiam haec altera forte commodior ad calculandum et conſtruendum: $b = \frac{\sqrt{(a^2 \, fin. \beta^2 \, fin. (\beta + \phi + \psi^2)}}{fin. \beta}$ $\frac{+ (c \, fin. \phi \pm a \, fin. \beta \, cos. (\beta + \phi + \psi))^2)}{fin. \beta}$.

Latus

Latus autem AB per hanc aequationem habetur: $a = \frac{\mp fin.\ \phi\ cos.\ (\beta+\phi+\psi) \pm \surd (b^2\ fin.\ \beta^2 - c^2\ fin.\ \phi^2\ fin.\ (\beta+\phi+\psi)^2)}{fin.\ \beta}$ et latus $AD = c = \frac{fin.\ \beta\ (\mp a\ cos.\ (\beta+\phi+\psi) \pm \surd (b^2 - a^2\ fin.\ (\beta+\phi+\psi)^2))}{fin.\ \phi}$.

Coroll. II.

§. 469. Ad inveniendum angulum ACD multiplico terminum $c^2\ fin.\ \phi^2$ per $fin.\ \beta^2 + cos.\ \beta^2 = 1$, ut aequatio respectu anguli β fiat homogenea, et facta sinuum separatione, transpositione et divisione habetur haec aequatio: $(b^2 - a^2 - c^2\ fin.\ \phi^2 + 2ac\ fin.\ \phi\ fin.\ (\phi+\psi))\ tang.\ \beta^2 - 2ac\ fin.\ \phi\ cos.\ (\phi+\psi)\ tang.\ \beta = c^2\ fin.\ \phi^2$, ex qua operatione absoluta, obtinetur haec aequatio: $tang.\ \beta = \frac{c\ fin.\ \phi\ (a\ cos.\ (\phi+\psi) \pm \surd (b^2 - (c\ fin.\ \phi - a\ fin.\ (\phi+\psi))^2))}{b^2 - a^2 - c^2\ fin.\ \phi^2 + 2ac\ fin.\ \phi\ fin.\ (\phi+\psi)}$

Ad angulum D obtinendum, multiplico sinistrum aequationis membrum per $fin.\ \phi^2 + cos.\ \phi^2$ ad aequationem respectu anguli ϕ homogeneam faciendam et facta separatione transpositione et divisione pervenio ad hanc aequationem $((b^2 - a^2)\ fin.\ \beta^2 - c^2 + 2ac\ fin.\ \beta\ fin.\ (\beta+\psi))\ tang.\ \phi^2 - 2ac\ fin.\ \beta\ cos.\ (\beta+\psi)\ tang.\ \phi = -(b^2-a^2)\ fin.\ \beta^2$, ex qua operatione ad finem perducta obtinetur haec aequatio: $tang.\ \phi = \frac{fin\beta\ (ac\ cos.\ (\beta+\phi) \pm \surd (b^2c^2 - (ac\ fin.\ (\beta+\psi) + (b^2-a^2)\ fin.\ \beta)^2))}{(b^2-a^2)\ fin.\ \beta^2 - c^2 + 2ac\ fin.\ \beta\ fin.\ (\beta+\psi)}$

Schol.

Scholion I.

§. 470. Poterant vero etiam ſine hoc artificio obtineri ſolutiones per aequationem quadraticam bis extractam, ſive aequationem quarti gradus, terminis dimenſionum imparium deficientibus, ſic autem aequationes operoſiores et ad praxin minus aptae prodiiſſent, quarum idcirco evolutionem ſtudio brevitatis omitto.

Coroll. III.

§. 471. Ad obtinendum angulum A fit,

$$\cos.(\beta+\varphi+\psi)=\frac{(b^2-a^2)\,\text{ſin.}\,\beta^2-c^2\,\text{ſin.}\,\varphi^2}{2ac\,\text{ſin.}\,\beta\,\text{ſin.}\,\varphi}$$

et conſequenter ipſe angulus

$$\psi=\text{ang.}\cos.\frac{(b^2-a^2)\,\text{ſin.}\,\beta^2-c^2\,\text{ſin.}\,\varphi^2}{2ac\,\text{ſin.}\,\beta\,\text{ſin.}\,\varphi}-\beta-\varphi.$$

Scholion II.

§. 472. Si angulus lateris AB cum diagonali ſit acutus, obtinet haec aequatio, ſi vero ſit obtuſus erit,

$$\psi=\text{ang.}\cos.\frac{c\,\text{ſin.}\,\varphi^2-(b^2-a^2)\,\text{ſin.}\,\beta^2}{2ac\,\text{ſin.}\,\beta\,\text{ſin.}\,\varphi}-\beta-\varphi.$$

Verum etiam altera ſolutio haberetur, ſinus a ſe invicem ſeparando, quae, cum operoſior ſit et ſolutio vulgaris trigonometrica huic tetragonometricae praeferenda, eam ſtudio brevitatis omitto.

Scholion III.

§. 473. Inter ſex caſus qui in aequatione continentur, duo ſunt Tetragonometriae proprii,

prii, qui tunc obtinent, quando ex latere AD vel angulo D inveniendo figura construenda proponitur, et in Geometria practica duo problemata suppeditant, satis utilia juxta et pulchra. Caeteri quatuor casus utrique methodo et trigonometricae et tetragonometricae subsunt, quorum solutiones dedi, quia ex aequatione haud difficulter sequuntur, et solutionibus vulgaribus trigonometricis haud ita multum cedunt.

Coroll. IV.

§. 474. Si anguli A et D ponantur simul sumti duobus rectis aequales, ut prodeat trapezium parallelarum basium AB, CD, aequatio pro latere BC in hanc mutatur:

$$b^2 = \frac{\sqrt{(a^2 \,\text{fin.}\, \beta^4 + (c \,\text{fin.}\, \varphi \mp a \,\text{fin.}\, \beta \,\text{cos.}\, \beta)^2)}}{\text{fin.}\, \beta}$$

verum aequatio pro latere AB in hanc transformatur: $a = \pm \,\text{fin.}\, \varphi \,\text{cot.}\, \beta \pm \sqrt{(b^2 - c^2 \,\text{fin.}\, \varphi^2)}$, et pro latere AD fit:

$$c = \frac{\text{fin.}\, \beta \,(\pm a \,\text{cos.}\, \beta \pm \sqrt{(b^2 - a^2 \,\text{fin.}\, \beta^2)})}{\text{fin.}\, \varphi},$$

pro angulo ACD fit:

$$\text{tang.}\, \beta = \frac{c \,\text{fin.}\, \varphi (-a \pm \sqrt{(b^2 - c^2 \,\text{fin.}\, \varphi^2)})}{b^2 - a^2 - c^2 \,\text{fin.}\, \varphi^2}.$$

Coroll. V.

§. 475. Si angulus A ponatur rectus vel etiam tribus rectis aequalis, cadente vertice A supra diagonalem BD, et trapezio inverso prodeunte,

deunte, aequatio pro latere BC in hanc mutatur:

$$b = \frac{\sqrt{(a^2 \sin\beta^2 \cos(\beta+\varphi)^2 + (c \sin\varphi \mp a \sin\beta \sin(\beta+\varphi))^2)}}{\sin\beta}$$

et pro latere AB habetur:

$$a = \frac{\pm \sin\varphi \sin(\beta+\varphi) \pm \sqrt{(b^2 \sin\beta^2 - c^2 \sin\varphi^2 \cos(\beta+\varphi)^2)}}{\sin\beta}$$

et pro latere AD

$$c = \frac{\sin\beta(\pm a \sin(\beta+\varphi) \pm \sqrt{(b^2 - a^2 \cos(\beta+\varphi)^2)})}{\sin\varphi}$$

pro angulo ACD $tang.\ \beta =$

$$= \frac{c \sin\varphi(\mp a \sin\varphi \pm \sqrt{(b^2 - (c \sin\varphi \mp a \cos\varphi)^2)})}{b^2 - a^2 - c^2 \sin\varphi^2 \pm 2ac \sin\varphi \cos\varphi}.$$

Denique pro angulo D habetur haec expressio: $tang.\ \varphi =$

$$= \frac{\sin\beta(\mp ac \sin\beta \pm \sqrt{(b^2c^2 - (b^2-a^2)\sin\beta \pm ac \cos\beta)^2)})}{(b^2-a^2)\sin\beta^2 - c^2 \pm 2ac \sin\beta \cos\beta}.$$

Coroll. VI.

§. 476. Si angulus D fit rectus vel etiam aequalis tribus rectis, cadente vertice D ad finistram diagonalis AC, et trapezio inverso prodeunte, aequatio pro latere BC in hanc mutatur:

$$b = \frac{\sqrt{(a^2 \sin\beta^2 \mp 2ac \sin\beta \sin(\beta+\psi) + c^2)}}{\sin\beta},$$

et pro latere AB

$$a = \frac{\pm \sin(\beta+\psi) \pm \sqrt{(b^2 \sin\beta^2 - c^2 \cos(\beta+\psi)^2)}}{\sin\beta},$$

X et

et pro latere AD

$$c = \frac{fin.\,\beta\,(\pm a\,fin.\,(\beta+\psi) \pm \sqrt{(b^2 - a^2\,cos.\,(\beta+\psi)^2)})}{fin.\,\varphi}$$

Pro angulo ACD

$$tang.\,\beta = \frac{-\,ac\,fin.\,\psi \pm \sqrt{(b^2 - (c - a\,cos.\,\psi)^2)}}{b^2 - a^2 - c^2 + 2ac\,cos.\,\psi}.$$

Denique pro angulo A

$$\psi = ang.\,fin.\,\frac{-\,(b^2 - a^2)\,fin.\,\beta^2 + c^2}{2ac\,fin.\,\beta} - \beta.$$

Coroll. VII.

§. 477. Si angulus ACD fit rectus, aequatio pro latere BC in hanc mutatur:

$$b = \sqrt{(a^2\,cos.\,(\varphi+\psi)^2 + (c\,fin.\,\varphi \mp a\,fin.\,(\varphi+\psi))^2)}$$

Pro latere AD habetur:

$$c = \frac{\mp a\,fin.\,(\varphi+\psi) \pm \sqrt{(b^2 - a^2\,cos.\,(\varphi+\psi)^2)}}{fin.\,\varphi}.$$

Pro latere AB habetur:

$$a = \pm fin.\,\varphi\,fin.\,(\varphi+\psi) \pm \sqrt{(b^2 - a^2\,cos.\,(\varphi+\psi)^2)},$$

pro angulo D

$$tang.\,\varphi = \frac{-\,ac\,fin.\,\psi \pm \sqrt{(b^2c^2 - (b^2 - a^2 + ac\,cos.\,\psi)^2)}}{b^2 - a^2 - c^2 + 2ac\,cos.\,\psi}.$$

Pro angulo A,

$$\psi = ang.\,fin.\,\frac{c^2\,fin.\,\varphi^2 + a^2 - b^2}{2ac\,fin.\,\varphi} - \varphi.$$

Coroll. VIII.

§. 478. Si angulus A fit aequalis duobus rectis, lateribus AB, AD in diagonalem BD, incidentibus et trapezio in triangulum BCD abeunte; fit pro latere BC

$b =$

$b = \frac{\sqrt{} (a^2 \sin. \beta^2 \sin. (\beta+\varphi)^2 + (c \sin \varphi \mp a \sin. \beta \cos. (\beta+\varphi))^2)}{\sin. \beta}$ et pro lateris BD segmento BE, $a = \frac{\pm \sin. \varphi \cos. (\beta+\varphi) \pm \sqrt{} (b^2 \sin. \beta^2 - c^2 \sin. \varphi^2 \sin. (\beta+\varphi)^2)}{\sin. \beta}$ et pro segmento DE, $c = \frac{\sin. \beta (\pm a \cos. (\beta+\varphi) \pm \sqrt{} (b^2 - a^2 \sin. (\beta+\varphi)^2))}{\sin. \varphi}$ et pro angulo ACD $tang. \beta = \frac{c \sin. \varphi (-a \cos. \varphi \pm \sqrt{} (b^2 - (a+c)^2 \sin. \varphi^2))}{b^2 - a^2 - (2a+c) c \sin. \varphi^2}$ et pro angulo D $tang. \varphi = \frac{\sin. \beta (-a c \cos. \beta \pm \sqrt{} (b^2 c^2 - (b^2 - a^2 + a c)^2 \sin. \beta^2))}{(b^2 - a^2 - 2ac) \sin. \beta^2 - c^2}$.

Scholion IV.

§. 479. Per hanc positionem solvuntur duo problemata ad triangula spectantia, quorum prius aliis methodis vix solvitur, et tum obtinet, si quaeratur angulus D, alterum si linea ED. In neutro enim partialium dantur nisi duo, nec in totali dantur nisi duo, latus BC et angulus oppositus, in casu posteriori, una cum angulo partiali ECD et segmento BE, in priori casu datur idem latus, idem angulus partialis et segmentum BE.

Scholion. V.

§. 480. Si latus AD cadat intra latus AB, quomodocunque trapezium ſit inverſum, valet haec aequatio: $b^2 \text{ſin.}\, \beta^2 = a^2 \text{ſin.}\, \beta^2 + c^2 \text{ſin.}\, \phi^2 \pm 2ac\, \text{ſin.}\, \beta\, \text{ſin.}\, \phi \cos.(\beta+\phi)-\psi)$. Si vero latus AB cadat intra latus AD, quomodocunque trapezium ſit inverſum, valet aequatio ſupra data, et conſequenter generalis aequatio eſt hujusmodi: $b^2 \text{ſin.}\, \beta^2 = a^2 \text{ſin.}\, \beta^2 + c^2 \text{ſin.}\, \phi^2 \pm 2ac\, \text{ſin.}\, \beta\, \text{ſin.}\, \phi \cos.(\beta+\phi\pm\psi)$.

Problema XXXIV.

Fig. XXXIV. §. 481. In figura quadrilatera propoſita $ABCD$ inter haec ſex: latus $AB=a$, $AD=c$, et angulos $A=\psi$, $B=\lambda$, $D=\phi$, $ACD=\beta$; aequationem invenire

Solutio.

Cum ſit per antecedentia diagonalis $AC=\frac{c\, \text{ſin.}\, \phi}{\text{ſin.}\, \phi}$ et $\text{ſin.}\, ACB=-\text{ſin.}\,(\beta+\lambda+\phi+\psi)$, in triangulo ſiniſtro ſtatim pervenio ad hanc analogiam: $AB : \text{ſin.}\, ACB = AC : \text{ſin.}\, B$, h. e. ſymbolis ſubſtitutis: $a : -\text{ſin.}(\beta+\lambda+\phi+\psi) = \frac{c\, \text{ſin.}\, \phi}{\text{ſin.}\, \beta} : \text{ſin.}\, \lambda$, et conſequenter, $a\, \text{ſin.}\, \beta\, \text{ſin.}\, \lambda = -c\, \text{ſin.}\, \phi\, \text{ſin.}\,(\beta+\lambda+\phi+\psi)$, Q. E. I.

Coroll. I.

§. 482. Ex hac aequatione ſtatim habentur latera, cum ſit: $a=\frac{-c\, \text{ſin.}\, \phi\, \text{ſin.}\,(\beta+\lambda+\phi+\psi)}{\text{ſin.}\, \beta\, \text{ſin.}\, \lambda}$

nec

nec non $c = \frac{-a\, fin.\, \beta\, fin.\, \lambda}{fin.\, \phi\, fin.\, (\beta + \lambda + \phi + \psi)}$ ſtatim etiam obtinetur angulus A, cum enim ſit: $fin.\ (\beta + \lambda + \phi + \psi) = \frac{-a\, fin.\, \beta\, fin.\, \lambda}{c\, fin.\, \phi}$ deſcendendo ad angulos erit:

$$\psi = ang.\, fin. \frac{-a\, fin.\, \beta\, fin.\, \lambda}{c\, fin.\, \phi} - \beta - \lambda - \phi.$$

Altera ſolutio obtinetur ſeparando ſinus et brevitatis gratia ponendo: $fin.\, \lambda = h$, $fin.\, \beta = k$, $fin.\, \phi = m$, $fin.\, (\lambda + \beta + \phi) = p$, $cos.\, (\lambda + \beta + \phi) = q$, $fin.\, \psi = x$, $cos.\, \psi = \sqrt{(1 - x^2)}$ ſic enim facta ſeparatione, ſubſtitutione et transpoſitione, habetur haec aequatio: $ahk + cmqx = -cmp\sqrt{(1 - x^2)}$ ex qua operatione abſoluta obtinetur haec aequatio: $x = \frac{-ahk \pm p\sqrt{(c^2 m^2 - a^2 h^2 k^2)}}{cm} = fin.\, \psi.$

Coroll. II.

§. 483. Pro angulo B obtinendo habetur haec aequatio: $tang.\, \lambda = \frac{-c\, fin.\, \phi\, fin.\, (\beta + \phi + \psi)}{a\, fin.\, \beta + c\, fin.\, \phi\, cos.\, (\beta + \phi + \psi)}$ vel etiam haec:

$tang.\, \lambda = \frac{-c\, fin.\, \phi\, tang.\, (\beta + \phi + \psi)}{c\, fin.\, \phi + a\, fin.\, \beta\, fec.\, (\beta + \phi + \psi)}$, ſimiliter pro obtinendo angulo ACD habetur haec plane ſimilis: $tang.\, \beta = \frac{-c\, fin.\, \phi\, fin.\, (\lambda + \phi + \psi)}{a\, fin.\, \lambda + c\, fin.\, \phi\, cos.\, (\lambda + \phi + \psi)}$ vel etiam haec:

$tang.\, \beta = \frac{-c\, fin.\, \phi\, tang.\, (\lambda + \phi + \psi)}{c\, fin.\, \phi + a\, fin.\, \lambda\, fec\, (\lambda + \phi + \psi)}$.

Coroll. III.

§. 484. Pro angulo D obtinendo, transfundo aequationem in hanc formam: $2a\,\text{ſin.}\,\beta\,\text{ſin.}\,\lambda + c\cos.(\beta+\psi) = c\cos.(\beta+\lambda+2\phi+\psi)$, unde statim consequor hanc solutionem:

$$\phi = \tfrac{1}{2}\text{ang.}\cos.\frac{2a\,\text{ſin.}\,\beta\,\text{ſin.}\,\lambda + c\cos.(\beta+\lambda+\psi)}{c} - \frac{\beta-\lambda-\psi}{2}$$

altera solutio obtinetur multiplicando sinistrum aequationis membrum per $\text{ſin.}\,\phi^2 + \cos.\,\phi^2$, ad aequationem respectu anguli ϕ homogeneam reddendam, et brevitatis gratia ponendo, $\text{ſin.}\,\lambda = h$, $\text{ſin.}\,\beta = k$, $\text{ſin.}(\beta+\lambda+\psi) = m$, $\cos.(\beta+\lambda+\psi) = n$, $\text{tang.}\,\phi = t$, quare separando, transponendo, dividendo et substituendo habetur, $(ahk+cn)t^2 + cmt = -ahk$, unde operatione absoluta habetur haec aequatio:

$$t = \frac{-cm \pm \sqrt{(c^2m^2 - 4ahk(cn+ahk)}}{2(cn+ahk)} = \text{tang.}\,\phi.$$

Scholion I.

§. 485. Si ad abbreviandum ponatur ut prius et praeterea $\text{ſin.}\,\phi = x$, $\cos.\,\phi = \sqrt{(1-x^2)}$; sine hoc artificio obtinetur solutio per aequationem quadraticam, signo radicali duplici adfectam, sive per aequationem quarti gradus, terminis imparium dimensionum exulantibus, nam separando, substituendo et transponendo, habetur haec aequatio abbreviata: $ahk + cnx^2 = -cmx\sqrt{(1-x^2)}$, ex qua operatione absoluta, obtinetur haec aequatio:

$$x = \frac{\sqrt{(c^2m^2 - 2ahkn \pm m\sqrt{(c^2m^2 - 4ahk(cn+ahk))}}}{\sqrt{c}}.$$

Schol.

Scholion II.

§. 486. Inter ſex caſus in aequatione contentos non niſi duo ſunt Tetragonometriae proprii, qui tunc obtinent, quando figura ex inventione angulorum in dextro triangulo conſtruenda proponitur, et in Geometria practica conſtituunt duo problemata nova, utilia et ſatis elegantia, caeteri quatuor caſus utrique methodo et tetragonometricae et trigonometricae ſubſunt. Solutiones vero tetragonometricae trigonometricis vix cedunt.

Coroll. IV.

§. 487. Si anguli A et D ponantur ſimul ſumti aequales duobus rectis, prodeunte ſic trapezio parallelarum baſium AB, DC, aequatio pro latere AB, in hanc abit: $a = \frac{c \text{ ſin. } \phi \text{ ſin. } (\beta + \lambda)}{\text{ſin. } \beta \text{ ſin. } \lambda}$, et latere AD, $c = \frac{a \text{ ſin. } \beta \text{ ſin. } \lambda}{\text{ſin. } \phi \text{ ſin. } (\beta + \lambda)}$, pro angulo B habetur: $\text{tang. } \lambda = \frac{c \text{ ſin. } \phi \text{ ſin. } \beta}{a \text{ ſin. } \beta - c \text{ ſin. } \phi \text{ cos. } \beta}$ $= \frac{c \text{ ſin. } \phi \text{ tang. } \beta}{a \text{ tang. } \beta - c \text{ ſin. } \phi}$, vel etiam denique $\text{tang. } \lambda = \frac{c \text{ tang. } \beta}{a \text{ tang. } \beta \text{ coſec. } \phi - c}$, pro angulo ACD habetur: $\text{tang. } \beta = \frac{c \text{ ſin. } \phi \text{ ſin. } \lambda}{a \text{ ſin. } \lambda - c \text{ ſin. } \phi \text{ cos. } \lambda}$ vel etiam $\text{tang. } \beta = \frac{c \text{ ſin. } \phi \text{ tang. } \lambda}{a \text{ tang. } \lambda - c \text{ ſin. } \phi} = \frac{c \text{ tang. } \beta}{a \text{ tang. } \lambda \text{ coſec. } \phi - c}$.

X 4 Coroll.

Coroll. V.

§. 488. Si anguli B et A, ſint ſimul ſumti aequales duobus rectis, prodeunte trapezio parallelarum baſium BC, AD, aequatio pro latere AB in hanc mutatur: $a = \frac{c\,fin.\,\phi\,fin.\,(\beta+\phi)}{fin.\,\beta\,fin.\,\lambda}$ pro latere AD fit: $c = \frac{a\,fin.\,\beta\,fin.\,\lambda}{c\,fin.\,\phi\,fin.\,(\beta+\phi)}$ pro angulo ACD: $tang.\beta = \frac{c\,fin.\,\phi^2}{a\,fin.\,\lambda - c\,fin.\,\phi\,cos.\,\phi}$ $= \frac{c\,tang.\,\phi}{a\,fin.\,\lambda\,fec.\,\phi\,cofec.\,\phi - c}$. Pro angulo D $\phi = \frac{1}{2}\,ang.\frac{-2\,a\,fin.\,\beta\,fin.\,\lambda + c\,cos.\,\beta - \beta}{c}\;\frac{}{2}$

Sed in aequatione quadratica fit, $m = -k$, et $n = -cos.\,\beta$, quae igitur in hanc paululum immutatur: $t = \frac{ck \pm \sqrt{(c^2k^2 - 4ahk(ahk \pm cn))}}{2(ahk - cn)}$.

Coroll. VI.

§. 489. Si anguli diagonaliter oppoſiti B et D, ſint aequales duobus rectis, prodeunte trapezio circulo inſcriptibile, pro latere AB habetur haec expreſſio: $a = \frac{c\,fin.\,(\beta+\psi)}{fin.\,\beta}$, conſequenter $c = \frac{a\,fin.\,\beta}{fin.\,(\beta+\psi)}$ et $\phi = ang.\,fin.\frac{a.\,fin.\,\beta}{c} - \beta$ ſed in aequatione quadratica fit, $h = m$, $p = -k$, $q = -cos.\,\beta$, conſequenter aequatio in hanc mutatur:

tatur: $x = \frac{k}{c}(a\,q \pm \sqrt{}(c^2 - a^2 k^2))$, denique pro angulo ACD habetur: $tang.\,\beta = \frac{c\,fin.\,\psi}{a - c\,cos.\,\psi}$ $= \frac{c\,tang.\,\psi}{a\,fec.\,\psi - c}$.

Coroll. VII.

§. 490. Si angulus A fit rectus vel etiam tribus rectis aequalis, cadente vertice A fupra diagonalem BD, et trapezio inverfo prodeunte, habetur pro latere AB haec expreffio: $a = \frac{\mp c\,fin.\,\varphi\,cos.\,(\beta+\lambda+\varphi)}{fin.\,\beta\,fin.\,\lambda}$, pro latere AD haec $c = \frac{\mp a\,fin.\,\beta\,fin.\,\lambda}{fin.\,\varphi\,cos.\,(\beta+\lambda+\varphi)}$. Pro angulo B habetur: $tang.\,\lambda = \frac{\mp c\,fin.\,\varphi\,cos.\,(\beta+\varphi)}{a\,fin.\,\beta \mp c\,fin.\,\varphi\,fin.\,(\beta+\varphi)}$ vel etiam $tang.\,\lambda = \frac{c\,fin.\,\varphi\,cos.\,(\beta+\varphi)}{c\,fin.\,\varphi\,fin.\,(\beta+\varphi \mp a\,fin.\,\beta}$ $= \frac{c\,fin.\,\varphi}{c\,fin.\,\varphi\,tang.\,(\beta+\varphi) \mp a\,fin.\,\beta\,fec.\,(\beta+\varphi)}$. Pro angulo ACD, $tang.\,\beta = \frac{c\,fin.\,\varphi\,cos.\,(\lambda+\varphi)}{c\,fin.\,\varphi\,fin.\,(\lambda+\varphi \mp a\,fin.\,\lambda}$ vel etiam,

$$tang.\,\beta = \frac{c\,fin.\,\varphi}{c\,fin.\,\varphi\,tang.\,(\lambda+\varphi) \mp a\,fin.\,\lambda\,fec.\,(\lambda+\varphi)},$$

denique pro angulo D, erit aequatio angularis $\varphi = \tfrac{1}{2}\,ang.fin.\frac{\mp 2a\,fin.\,\beta\,fin.\,\lambda + c\,fin.\,(\beta+\lambda)}{c} \frac{-\beta-\lambda}{2}$

 Sed

Sed in aequatione quadratica fit, $m = \pm cos.(\beta+\lambda)$ $n = \mp fin.(\beta+\lambda)$, quae igitur in hanc paululum mutatur: $t = \frac{\mp cm \pm \sqrt{(c^2m^2 - 4ahk(ahk \pm cn))}}{2(ahk \pm cn)}$.

Coroll. VIII.

§. 491. Si angulus D fit rectus vel etiam aequalis duobus rectis, cadente vertice D ad finiftram diagonalis AC, et trapezio inverfo prodeunte, fit pro latere AB $a = \frac{-c\ cos.(\beta+\lambda\ \psi)}{fin.\beta\ fin.\lambda}$ et latere AD, $c = \frac{-a\ fin.\beta\ fin.\lambda}{cos.\ (\beta+\lambda+\psi)}$. Pro angulo A aequatio angularis: $\psi = ang.\ cos. \frac{-a\,fin.\beta\,fin.\lambda}{c} - \beta - \lambda$;

Sed in aequatione quadratica erit, $m = \pm 1$, $p = \pm cos.(\lambda+\beta)$, $q = \mp fin.(\lambda+\beta)$, quae igitur in hanc paululum mutatur: $x = \frac{ahkq + p\sqrt{(c^2 - a^2h^2k^2)}}{c}$;

pro angulo B fit, $tang.\ \lambda = \frac{-c\,cos.(\beta+\psi)}{a\,fin.\beta - c\,fin.(\beta+\psi)}$

vel etiam $tang.\ \lambda = \frac{-c}{a\,fin.\beta\,fec.(\beta+\psi) - c\,tang.(\beta+\psi)}$

denique pro angulo ACD

$$tang.\ \beta = \frac{-c\ cos.\ (\lambda+\psi)}{a\ fin.\ \lambda - c\ fin.\ (\lambda+\psi)}$$

$$= \frac{-c}{a\ fin.\ \lambda\ fec.\ (\lambda+\psi) - c\ tang.\ (\lambda+\psi)}.$$

Coroll. IX.

§. 492. Si angulus B ſit rectus vel etiam tribus rectis aequalis, cadente B ad dextram diagonalis AC, et trapezio inverſo prodeunte, pro latere AB habetur haec expreſſio:

$a = \frac{-c \, \text{ſin.}\, \varphi \, \cos.(\beta+\varphi+\psi)}{\text{ſin.}\, \beta}$, pro latere AD

$c = \frac{-a \, \text{ſin.}\, \beta}{\text{ſin.}\, \varphi \, \cos.(\beta+\varphi+\psi)}$, pro angulo A ſit aequatio angularis: $\psi = \text{ang.}\cos.\frac{-a\,\text{ſin.}\,\beta}{c\,\text{ſin.}\,\varphi} - \beta - \varphi$,

ſed in aequatione quadratica ſit, $h = \pm 1$, $p = \pm \cos.(\beta+\varphi)$, $q = \mp \text{ſin.}(\beta+\varphi)$, quae igitur in hanc paululum mutatur:

$$x = \frac{a\,k\,q + p\,\sqrt{(c^2 m^2 - a^2 k^2)}}{c\,m} = \text{ſin.}\,\psi,$$

Pro angulo ACD habetur haec aequatio:

$$\text{tang.}\,\beta = \frac{c\,\text{ſin.}\,\varphi\,\cos.(\varphi+\psi)}{c\,\text{ſin.}\,\varphi\,\text{ſin.}(\varphi+\psi) \mp c}$$

$$= \frac{c}{c\,\text{tang.}(\varphi+\psi) \mp c\,\text{ſec.}(\varphi+\psi)\,\text{coſec.}\,\varphi}$$

denique pro angulo D habetur:

$$\varphi = \tfrac{1}{2}\,\text{ang.ſin.}\frac{-2a\,\text{ſin.}\,\beta + c\,\text{ſin.}(\beta+\psi)}{c} - \beta - \psi.$$

In aequatione quadratica autem ſit, $h = \pm 1$, $m = \pm \cos.(\beta+\psi)$, $n = \mp \text{ſin.}(\beta+\psi)$, et prodeinde aequatio in hanc abit:

$$t = \frac{-cm + \sqrt{(c^2 m^2 - 4ak(ak - cn))}}{2(ak - cn)}.$$

Coroll.

Coroll. X.

§. 493. Si angulus A ſit aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus, et trapezio in triangulum abeunte, fit pro latere AB,

$a = \frac{c\, fin.\, \phi\, fin.\, (\beta+\lambda+\phi)}{fin.\, \beta\, fin.\, \lambda}$ hinc pro latere $A D$,

$c = \frac{a\, fin.\, \beta\, fin.\, \lambda}{fin.\, \phi\, fin.\, (\beta+\lambda+\phi)}$, pro angulo B,

$tang.\, \lambda = \frac{c\, fin.\, \phi\, fin.\, (\beta+\phi)}{a\, fin.\, \beta - c\, fin.\, \phi\, cos.\, (\beta+\phi)}$ vel

etiam haec $tang.\, \lambda = \frac{c\, tang.\, (\beta+\phi)}{a\, fin.\, \beta\, fec.\, (\beta+\phi)\, cofec.\, \phi - c}$

pro angulo ACD

$tang.\, \beta = \frac{c\, fin.\, \phi\, fin.\, (\lambda+\phi)}{a\, fin.\, \lambda - c\, fin.\, \phi\, cos.\, (\lambda+\phi)}$

$= \frac{c\, tang.\, (\lambda+\phi)}{a\, fin.\, \lambda\, fec.\, (\lambda+\phi)\, cofec.\, \phi - c}$ denique pro angulo D, habetur haec aequatio angularis:

$\phi = \frac{1}{2}\, ang.\, cos. \frac{-2a\, fin.\, \lambda\, fin.\, \beta + c\, cos.\, (\beta+\lambda)}{c} - \frac{\beta - \lambda}{2}.$

Scholion III.

§. 494. Hac poſitione ultimi Corollarii ſolvuntur duo problemata ad triangula ſpectantia, quorum prius aliis methodis non ſolvitur, quod tunc obtinet ſi angulus D; alterum, ſi ED ſegmentum lateris BD quaeratur, nam in triangulis partialibus non dantur niſi duo,

nec

nec in totali nisi duo, et in uno casu angulus partialis ECD. In altero casu etiam duo, segmentum BE et idem angulos partialis. Caeteri casus etiamsi nova quadam ratione quoque solvantur, tamen etiam trigogonometrice expediuntur,

Scholion IV.

§. 495. Si latus AD cadat intra latus AB, sive trapezium sit partialiter sive totaliter inversum, valet aequatio haec: $a\,\textit{sin.}\,\lambda\,\textit{sin.}\,\beta = c\,\textit{sin.}\,\varphi\,\textit{sin.}\,(\beta+\varphi-\lambda-\psi)$, si vero latus AB cadat intra latus AD, sive trapezium sit totaliter sive partialiter inversum, valet haec aequatio: $a\,\textit{sin.}\,\beta\,\textit{sin.}\,\lambda = c\,\textit{sin.}\,\varphi\,\textit{sin.}\,(\beta+\varphi+\psi-\lambda)$, consequenter est in hoc casu aequatio generalissima: $a\,\textit{sin.}\,\beta\,\textit{sin.}\,\lambda = \mp c\,\textit{sin.}\,\varphi\,\textit{sin.}\,(\beta+\varphi \pm f \pm \lambda)$.



CAPUT

CAPUT X.

Continens tria problemata quartae classis particularis, sub posteriore principali contentae.

Problema XXXV.

Fig. XXXV. §. 496. In proposita figura quadrilatera $ABCD$, inter haec sex: latus $BC=b$, $CD=d$, angulum $A=\psi$, $B=\lambda$, $D=\varphi$, $ACD=\beta$, aequationem invenire

Solutio.

Cum sit per antecedentia $AC=\frac{d\,fin.\varphi}{fin.(\psi+\varphi)}$, et $fin.\ CAB=-fin.(\beta+\varphi+\psi)$; in triangulo sinistro statim pervenio ad hanc analogiam $BC : fin.\ CAB=AC : fin.\ B$, h. e. symbolis substitutis $b : -fin.(\beta+\varphi+\psi)=\frac{d\ fin.\ \varphi\ 1}{fin.\ (\beta+\varphi)} : fin.\ \lambda$ et consequenter habetur haec aequatio: $b\,fin.(\beta+\varphi)\ fin.\ \lambda=-d\ fin.\ \varphi\ fin.\ (\beta+\varphi+\psi)$, Q. E. I.

Coroll. I.

§. 497. Ex hac aequatione statim in ipsos oculos incurrit esse:

latus $BC=b=\frac{-d\ fin.\ \varphi\ fin.\ (\beta+\varphi+\psi)}{fin.(f+\varphi)\ fin.\lambda}$, et

latus $CD=d=\frac{-b\ fin.\ \lambda\ fin.\ (\beta+\varphi)}{fin.\ \varphi\ fin.\ (\beta+\varphi+\psi)}$,

nec

nec non anguli B $fin.\ \lambda = \frac{-d\, fin.\ \phi\ fin.(\beta+\phi+\psi)}{b\, fin.(\beta+\phi)}$

et cum fit $fin.(\beta+\phi+\psi) = \frac{-b\ (\beta+\phi)\ fin.\ \lambda)}{d\, fin.\ \phi}$

erit angulus ipfe,

$$\psi = ang.\, fin. \frac{-b\ fin.\ \lambda\ fin.\ (\beta+\phi)}{c\ fin.\ \phi} - \beta - \phi.$$

Altera folutio obtinetur finus a fe invicem feparando; nam pofito ad abbreviandum $fin.\ \lambda = h$, $fin.\ \phi = k$, $tang.\ (\beta+\phi) = t$, $fin.\ \psi = x$, $cos.\ \psi = \surd(1-x^2)$ et factis fubftitutione et transpofitione prodit haec aequatio: $bht + dkx = -dkt\surd(1-x^2)$ ex qua obtinetur:

$$x = \frac{t(-bh \pm \surd(d^2k^2(1+t^2) - b^2h^2t^2))}{dk(1+t^2)} = fin.\ \psi.$$

Coroll. II.

§. 498. Pro obtinendo angulo ACD, ad hanc aequationem pervenio:

$$tang.\ (\beta+\phi) = \frac{-d\ fin.\ \phi\ fin.\ \psi}{b\ fin.\ \lambda + d\ fin.\ \phi\ cos.\ \psi}$$

$= \frac{-d\ fin.\ \psi}{d\ cos.\ \psi + b\ fin.\ \lambda\ cofec.\ \phi}$, hinc ad ipfum angulum defcendendo obtinetur:

$$\beta = ang.\ tang. \frac{-d\ fin.\ \psi}{d\ cos.\ \psi - b\ fin.\ \lambda\ cofec.\ \phi} - \phi.$$

Ad obtinendum angulum D, finibus a fe invicem feparatis, et pofito ad abbreviandum $fin.\ \lambda = h$, $fin.\ \beta = m$, $cos.\ \beta = n$, $fin.(\beta+\psi) = p$, $cos.(\beta+\psi) = q$, et $nq + mp \pm s = cos.\ \psi$, $fin.\ \phi = x$, $cos.\ \phi = \surd(1-x^2)$, facta fubftitutione et transpofitio-

positione habetur: $bhnx+dqx^2=-(bhm+dpx)$ $\surd(1-x^2)$, ex qua operatione absoluta ad hanc aequationem quarti gradus pervenio:

$$x^4+\frac{2bhsx^3}{d}+\frac{(b^2h^2-p^2)x^2}{d^2}-\frac{2bhmpx}{d}-\frac{b^2h^2m^2}{d^2}=0.$$

Scholion I.

§. 499. Inter sex casus qui in aequatione implicantur, duo sunt Tetragonometriae proprii, qui tunc obtinent, quando figura ex angulis in triangulo dextro inveniendis, construenda proponitur, qui casus in Geometria practica problemata nova, utilia et satis pulchra suppeditant. Caeteri quatuor casus utrique methodo et tetragonometricae et trigonometricae subsunt, sed ita tamen ut solutiones tetragonometricae, trigonometricis non multum cedunt.

Coroll. III.

§. 500. Si anguli A et D simul sumti sint aequales duobus rectis, prodeunte trapezio parallelarum basium AB, CD, aequatio pro latere CB in hanc mutatur: $b=\frac{d\, \text{sin.}\, \varphi\, \text{sin.}\, \beta}{\text{sin.}\, \lambda\, \text{sin.}\, (\beta+\varphi)}$, vel etiam angulo φ eliminato: $b=\frac{d\, \text{sin.}\, \psi\, \text{sin.}\, \beta}{\text{sin.}\, \lambda\, \text{sin.}\, (\beta-\psi)}$ et pro latere DC, $d=\frac{b\, \text{sin.}\, \lambda\, \text{sin.}\, (\beta+\varphi)}{\text{sin.}\, \varphi\, \text{sin.}\, \beta}$ $=\frac{b\, \text{sin.}\, \lambda\, \text{sin.}\, (\beta-\psi)}{\text{sin.}\, \psi\, \text{sin.}\, \beta}$ pro angulo B habetur:

sin.

$$\text{fin.}\ \lambda = \frac{d\,\text{fin.}\ \psi\ \text{fin.}\ \beta}{b\ \text{fin.}\ (\beta - \psi)} = \frac{d\ \text{fin.}\ \phi\ \text{fin.}\ \beta}{b\ \text{fin.}\ (\beta + \psi)},$$ pro angulo ACD fit, angulo ψ eliminato:

$$\text{tang.}\ (\beta + \phi) = \frac{-d\ \text{fin.}\ \phi^2}{b\ \text{fin.}\ \lambda - d\ \text{fin.}\ \phi\ \text{cos.}\ \phi},$$ vel etiam

$$\text{tang.}\ (\beta + \phi) = \frac{-d\ \text{tang.}\ \phi}{b\ \text{fin.}\ \lambda\ \text{fec.}\ \phi\ \text{cofec.}\ \phi - d},$$ et hinc angulus ipſe, $\beta = \text{ang.tang.} \frac{d\ \text{tang.}\ \phi}{d - b\ \text{fin.}\ \lambda\ \text{fec.}\ \phi\ \text{cofec.}\ \phi} - \phi$.

Coroll. IV.

§. 501. Si anguli A et D ponantur ſimul ſumti aequales duobus rectis, prodeunte trapezio parallelarum baſium AD, BC, aequatio pro latere BC in hanc mutatur, angulo λ eliminato: $b = \frac{-d\ \text{fin.}\ \phi\ \text{fin.}\ (\beta + \phi + \psi)}{\text{fin.}\ \phi\ \text{fin.}\ (\beta + \phi)}$,
ϕ autem eliminato habetur:

$$b = \frac{-d\ \text{fin.}\ \phi\ \text{fin.}\ (\beta + \phi - \lambda)}{\text{fin.}\ \lambda\ \text{fin.}\ (\beta + \phi)},$$ pro latere DC

$$d = \frac{-b\ \text{fin.}\ \psi\ \text{fin.}\ (\beta + \phi}{\text{fin.}\ \phi\ \text{fin.}\ (\beta + \phi + \psi)} = \frac{b\ \text{fin.}\ \lambda\ \text{fin.}\ (\beta + \phi)}{\text{fin.}\ \phi\ \text{fin.}\ (\beta + \phi - \lambda)}$$

pro angulo ACD invenitur, eliminato ψ,

$$\text{tang.}\ (\beta + \phi) = \frac{-d\ \text{fin.}\ \phi\ \text{fin.}\ \lambda}{b\ \text{fin.}\ \lambda - d\ \text{fin.}\ \phi\ \text{cos.}\ \lambda},$$ vel etiam

$$\text{tang.}\ (\beta + \phi) = \frac{-d\ \text{fin.}\ \phi\ \text{tang.}\ \lambda}{b\ \text{tang.}\ \lambda - d\ \text{fin.}\ \phi},$$ et ad angulum ipſum deſcendendo habetur:

$$\beta = \text{ang. tang.} \frac{-d\ \text{fin.}\ \phi\ \text{tang.}\ \lambda}{b\ \text{tang.}\ \lambda - d\ \text{fin.}\ \phi} - \phi.$$ In aequa-

 tione

tione quarti gradus pro angulo D fit, $h = fin.\,\varphi = fin.\,\lambda$, et $s = -cos.\,\lambda$, et hinc fecundus aequationis terminus figno negativo afficiendus; caeterum aequatio quoad formam externam invariata manet.

Coroll. V.

§. 502. Si anguli diagonaliter oppofiti B et D, ponantur fimul fumti aequales duobus rectis, prodeunte trapezio circulo infcriptibili, aequatio pro latere BC in hanc mutatur:
$b = \frac{-d\,fin.\,(\beta+\varphi+\psi)}{fin.\,(\beta+\varphi)}$, angulo λ eliminato, fed φ eliminato erit, $b = \frac{-d\,fin.\,(\beta+\psi-\lambda)}{fin.\,(\beta-\lambda)}$ et pro latere CD, $d = \frac{-b\,fin.\,(\beta+\varphi)}{fin.\,(\beta+\varphi+\psi)} = \frac{-b\,fin.\,(\beta-\lambda)}{fin.\,(\beta+\psi-\lambda)}$, pro angulo A erit aequatio angularis:
$\psi = ang\,fin.\frac{-b\,fin.\,(\beta+\varphi)}{d} - \beta - \varphi$, vel etiam
$\psi = ang.\,fin.\frac{-b\,fin.\,(\beta-\lambda)}{d} + \lambda - \beta$, fed in aequatione quadratica fit $h = k$, quae igitur in hanc paululum mutatur:
$x = \frac{t(-b \pm \surd(d^2(2+t^2) - b^2t^2))}{d(1+t^2)}$ pro angulo ACD habetur: $tang.\,(\beta+\varphi) = \frac{-d\,fin.\,\psi}{b+d\,cos.\,\psi}$, vel etiam $tang.\,(\beta+\varphi) = \frac{-d\,tang.\,\psi}{d+b\,fec.\,\psi}$ et hinc ipfe angulus erit: $\beta = ang.\,tang.\frac{-d\,tang.\,\psi}{d+b\,fec.\,\psi} - \varphi$.

Coroll.

Coroll. VI.

§. 503. Si angulus A ſit rectus vel etiam aequalis tribus rectis, cadente A ſupra diagonalem BD, et trapezio inverſo prodeunte, fit aequatio pro latere BC, $b = \frac{\pm d \, \text{ſin.}\, \varphi}{\text{ſin.}\, \lambda \, \text{tang.}\, (\beta + \varphi)}$, pro latere DC, $d = \frac{\mp b \, \text{ſin.}\, \lambda \, \text{tang.}\, (\beta + \varphi)}{\text{ſin.}\, \varphi}$, pro angulo B erit, $\text{ſin.}\, \lambda = \frac{\mp d \, \text{ſin.}\, \varphi}{b \, \text{tang.}\, (\beta + \varphi)}$, pro angulo ACD $\text{tang.}\, (\beta + \varphi) = \frac{\mp d \, \text{ſin.}\, \varphi}{b \, \text{ſin.}\, \lambda}$, et hinc ipſe angulus $\beta = \text{ang. tang.} \frac{\mp d \, \text{ſin.}\, \varphi}{b \, \text{ſin.}\, \lambda} - \varphi$, denique pro angulo D erit in aequatione quarti gradus: $p = \pm \cos.\, \beta = \pm n$, $q = \mp \text{ſin.}\, \beta = \mp m$, et, $s = o$, quae igitur in hanc abit:

$$x^4 + \frac{(b^2 h^2 - n^2)}{d^2} x^2 \frac{\mp 2bhmn\, x}{d} - \frac{b^2 h^2 m^2}{d^2} = o.$$

Coroll. VII.

§. 504. Si angulus D ſit rectus vel etiam aequalis tribus rectis, cadente D, ad ſiniſtram diagonalis AC, et trapezio inverſo prodeunte, expreſſio lateris BC in hanc mutatur:

$b = \frac{\mp d \cos.\, (\beta + \psi)}{\cos.\, \beta \, \text{ſin.}\, \lambda}$, et conſequenter,

$d = \frac{\mp b \cos.\, \beta \, \text{ſin.}\, \lambda}{\cos.\, (\beta + \psi)}$ et $\text{ſin.}\, \lambda = \frac{\mp d \cos.\, (\beta + \psi)}{b \cos.\, \beta}$

 et

et $\psi = ang.\ cos. \frac{\mp b\ cos.\ \beta\ fin.\ \lambda}{d} - \beta$, fed in aequatio quadratica fit, $k = \pm 1$, et $t = -\ cot.\ \beta$, quae igitur in hanc mutatur:

$x = \frac{t(\pm bh \pm \sqrt{(d^2(1+t^2) - b^2h^2t^2)})}{d(1+t^2)}$; pro angulo ACD habetur: $cot.\ \beta = \frac{\pm d\ fin.\ \psi}{b\ fin.\ \lambda \pm d\ cos.\ \psi}$ vel etiam $tang.\ \beta = \frac{b\ fin.\ \lambda \pm d\ cos.\ \psi}{\mp d\ fin.\ \varphi}$ et angulus ipfe, $\beta = ang.\ tang. \frac{b\ fin.\ \lambda \pm d\ cos.\ \psi}{\mp d\ fin.\ \varphi}$.

Coroll. VIII.

§. 505. Si angulus B fit rectus vel etiam tribus rectis aequalis, cadente B ad dextram diagonalis, et trapezio inverfo prodeunte, aequatio pro latere B in hanc mutatur:

$b = \frac{\mp d\ fin.\ \varphi\ fin.\ (\beta + \varphi + \psi)}{fin.\ (\beta + \varphi)}$ et pro latere DC in hanc: $d = \frac{\mp b\ fin.\ (\beta + \varphi)}{fin.\ \varphi\ fin.\ (\beta + \varphi + \psi)}$. Pro angulo A fit aequatio angularis:

$\psi = ang.\ fin. \frac{\mp b\ fin.\ (\beta + \varphi)}{d\ fin.\ \varphi} - \beta - \varphi$.

Sed in aequatione quadratica fit, $h = \pm 1$, quae igitur in hanc paululum mutatur:

$x = \frac{t(\mp b \pm \sqrt{(d^2h^2(1+t^2) - b^2t^2)})}{dk(1+t^2)} = fin.\ \psi$.

Pro

Pro angulo $A\,C\,D$ fit:

$tang.\,(\beta+\phi) = \frac{-\,d\ sin.\ \phi\ sin.\ \psi}{d\ sin.\ \phi\ \ cos.\ \psi \pm b}$ vel etiam

$tang.\,(\beta+\phi) = \frac{-\,d\ tang.\ \psi}{d \pm b\ sec.\ \psi\ \ cosec.\ \phi}$. In aequatione quarti gradus pro angulo D fit $h = \pm 1$, quae igitur in hanc paululum mutatur:

$$x^4 \pm \frac{2bsx}{d} + \frac{(b^2}{d^2}\,p^2)\,x^2 \mp \frac{2bmpx}{d} - \frac{bm^2}{d^2} = 0.$$

Coroll. IX.

§. 506. Si angulus ACD ponatur rectus; aequatio pro latere BC in hanc mutatur:

$$b = \frac{-\,d\ sin.\ \phi\ \ cos.\ (\phi+\psi)}{cos.\ \phi\ \ sin.\ \lambda} = \frac{-\,d\ tang.\ \phi\ cos.\ (\phi+\psi)}{sin.\ \lambda},$$

pro latere $C\,D$, $d = \frac{-\,b\ sin.\ \beta}{tang.\ \phi\ \ cos.\ (\phi+\psi)}$, pro angulo A fit aequatio angularis:

$\psi = ang.\,cos.\ \frac{-\,b\ sin.\ \lambda}{d\ tang\ \phi} - \phi$. Sed in aequatione quadratica fit, $t = -\,cot.\ \phi$, et consequenter aequatio in hanc paululum mutatur:

$x = \frac{t\,(bh \mp \surd\,(d^2k^2(1+t^2) - bh^2t^2))}{dk\,(1-t^2)}$; pro angulo B est $sin.\,\lambda = \frac{-\,d\ tang.\ \phi\ \ cos.\ (\phi+\psi)}{b}$; in aequatione quarti gradus pro angulo D fit, $m = 1$, $p = cos.\,\psi$, $q = -\,sin.\,\psi$, quae igitur in hanc paululum mutatur:

$$x^4 + \frac{2bhsx^3}{d^2} + \frac{(b^2h^2}{d^2} - p^2)\,x^2 - \frac{2bhpx}{d} - \frac{b^2h^2}{\cdot} = 0.$$

 Schol.

Scholion II.

§. 507. Si latus AD, cadat intra latus BD, ſive trapezium ſit partialiter, ſive totaliter inverſum, valet haec aequatio: $b\,\mathit{ſin.}\,\lambda\,\mathit{ſin.}\,(\beta+\varphi)=d\,\mathit{ſin.}\,\varphi\,\mathit{ſin.}\,(\beta+\varphi-\psi)$; Si vero latus AB cadat intra latus AD, ſive trapezium ſit partialiter ſive totaliter inverſum, valet haec aequatio: $b\,\mathit{ſin.}\,\lambda\,\mathit{ſin.}\,(\beta+\varphi)=d\,\mathit{ſin.}\,\varphi\,\mathit{ſin.}\,(\beta+\varphi+\psi)$, conſequenter eſt aequatio generalis: $b\,\mathit{ſin.}\,\lambda\,\mathit{ſin.}\,(\beta+\varphi)=\mp d\,\mathit{ſin.}\,\varphi\,\mathit{ſin.}\,(\beta+\varphi\pm\psi)$.

Problema XXXVI.

Fig. XXXVI. §. 508. In figura quadrilatera propoſita $ABCD$, inter haec ſex: latus $AB=a$, $BC=b$, $CD=d$, angulum $A=\psi$, $ACD=\beta$, $D=\varphi$; aequationem invenire

Solutio.

Cum ſit per antecedentia $AC=\dfrac{d\,\mathit{ſin.}\,\varphi}{\mathit{ſin.}\,(\beta+\varphi)}$ et per Problem. XXXIII. ſolutionem $\mathit{cos.}\,CAB=-\mathit{cos.}\,(\beta+\varphi+\psi)$, et ab angulo B in diagonalem demiſſo perpendiculo Bb, ſegmentum $Ab=-a\,\mathit{cos.}\,(\beta+\varphi+\psi)$; in triangulo ſiniſtro per Theorema (Eucl. II. 11.) $BC^2=Ab^2+AC^2-2\,AC\,Ab$, conſequenter ſymbolis ſubſtitutis ad hanc aequationem pervenio: $(b^2-a^2)\,\mathit{ſin.}\,(\beta+\varphi)^2=d^2\,\mathit{ſin.}\,\varphi^2+2\,ad\,\mathit{ſin.}\,\varphi\,\mathit{ſin.}\,(\beta+\varphi)\,\mathit{cos.}\,(\beta+\varphi+\psi)$. Q. E. I.

Coroll.

Coroll. I.

§. 509. Ex hac aequatione facillime obtinetur latus BC, cum sit:

$$b = \frac{\surd\left(a^2 \sin.(\beta+\phi)^2 \pm 2ad \sin.\phi \sin.(\beta+\phi) \cos.(\beta+\phi+\psi) + d^2 \sin.\phi^2\right)}{\sin.(\beta+\phi)}$$

vel etiam haec aptior ad calculandum et construendum:

$$b = \frac{\surd\left(a^2 \sin.(\beta+\phi)^2 \sin.(\beta+\phi+\psi)^2 + (d \sin.\phi \pm a \sin.(\beta+\phi) \cos.(\beta+\phi+\psi))^2\right)}{\sin.(\beta+\phi)}$$

et latus AB per hanc aequationem:

$$a = \frac{-d \sin.\phi \cos.(\beta+\phi+\psi) \pm \surd\left(b^2 \sin.(\beta+\phi) - d^2 \sin.\phi^2 \sin.(\beta+\phi+\psi)^2\right)}{\sin.(\beta+\phi)}.$$

Denique latus tertium CD per hanc aequationem exprimitur:

$$d = \frac{\sin.(\beta+\phi)\left(-a \cos.(\beta+\phi+\psi) \pm \surd(b^2 - a^2 \sin.(\beta+\phi+\psi)^2)\right)}{\sin.\phi}.$$

Coroll. II.

§. 510. Ex hac aequatione etiam faciliter obtinetur angulus A, cum sit:

$$\cos.(\beta+\phi+\psi) = \frac{(b^2-a^2)\sin.(\beta+\phi)^2 - d^2 \sin\phi^2}{2ad \sin.\phi \sin.(\beta+\phi)}$$

atque hinc ad ipsos angulos descendendo habetur haec aequatio:

$$\psi = \text{ang. cos.} \frac{(b^2-a^2)\sin.(\beta+\phi)^2 - d^2 \sin.\phi^2}{2ad \sin.\phi \sin.(\beta+\phi)} - \beta - \phi.$$

 Pro

Pro obtinendo angulo ACD multiplico $d^2 \mathit{fin.}\,\phi^2$ per $\mathit{fin.}(\beta+\phi)^2 + \mathit{cos.}(\beta+\phi)^2 = 1$, ad omnes terminos refpectu anguli β homogeneos reddendos et facta feparatione, transpofitione et divifione proveniet haec aequatio: $(b^2 - a^2 + 2ad\,\mathit{fin.}\,\phi\,\mathit{fin.}\,\psi - d^2\,\mathit{fin.}\,\phi^2)\,\mathit{tang.}(\beta+\phi)^2 - 2ad\,\mathit{fin.}\,\phi\,\mathit{cos.}\,\psi\,\mathit{tang.}(\beta+\phi) = d^2\,\mathit{fin.}\,\phi^2$, ex qua operatione ad finem perducta obtinetur haec aequatio:

$$\mathit{tang.}\,(\beta+\phi) = \\ = \frac{d\,\mathit{fin.}\,\phi\,(a\,\mathit{cos.}\,\psi \pm \surd(b^2 - (d\,\mathit{fin.}\,\phi - a\,\mathit{fin.}\,\psi)^2))}{b^2 - a^2 + 2ad\,\mathit{fin.}\,\phi\,\mathit{fin.}\,\psi - d^2\,\mathit{fin.}\,\phi^2}$$

confequenter ad angulum defcendendo habetur:

$$\beta = \mathit{ang.\ tang.}\ \frac{d\,\mathit{fin.}\,\phi\,(a\,\mathit{cos.}\,\psi \pm \surd(b^2 - (d\,\mathit{fin.}\,\phi - a\,\mathit{fin.}\,\psi)^2))}{b^2 - a^2 + 2ad\,\mathit{fin.}\,\mathit{fin.}\,\psi - d\,\mathit{fin.}\,\phi^2} - \phi.$$

Scholion I.

§. 511. Inter fex cafus, qui in aequatione continentur, tres dantur Tetragonometriae proprii, qui totidem in Geometria practica problemata nova, utilia et fatis pulchra fuppeditant, qui obtinent, quando ex aliquo trium in dextro triangulo inveniendo figura conftruenda proponitur, horum cafuum duos folutos dedi, tertium quando angulus D quaeritur, nondum utiliter folvere potui, cum aequatio ad fextum gradum afcendat, et magna conftet multitudine terminorum, a cujus igitur evolutione, ut ad praxin parum utilis, abftinui et ftudio brevitatis omifi. Caeteri tres cafus utrique methodo et tetragonometricae et trigonometricae fubfunt, fed ita ut trigonometrice fere facilius expediantur, quos tamen cafus evolvi, cum ex aequatione facile eruantur.

Coroll.

Coroll. III.

§. 512. Si anguli A et D ſimul ſumti ſint aequales duobus rectis, prodeunte ſic trapezio parallelarum baſium AB, CD, aequatio pro latere BC in hanc mutatur:

$$b = \frac{\sqrt{}(a^2 \text{fin.} \beta^2 \text{fin.} (\beta+\varphi)^2 + (d \text{fin.} \varphi \pm a \cos. \beta \text{fin.} (\beta+\varphi)^2))}{\text{fin.} (\beta+\varphi)},$$

ſed pro latere AB habetur haec:

$$a = \frac{d \text{fin.} \varphi \cos. \beta \pm \sqrt{}(b^2 \text{fin.} (\beta+\varphi) - d^2 \text{fin.} \varphi^2 \text{fin.} \beta^2)}{\text{fin.} (\beta+\varphi)}$$

pro latere CD,

$$d = \frac{\text{fin.} (\beta+\varphi) (a \cos. \beta \pm \sqrt{}(b^2 - a^2 \text{fin.} \beta^2))}{\text{fin.} \varphi},$$

aequatio pro angulo ACD in hanc mutatur, φ eliminato:

$$\text{tang.} (\beta+\varphi) = \frac{d \text{fin.} \psi (a \cos. \psi \pm \sqrt{}(b^2 - (d-a)^2 \text{fin.} \psi^2)}{b^2 - a^2 + (2a-d) d \text{fin.} \psi^2}$$

et angulus ipſe,

$$\beta = \text{ang.tang.} \frac{d \text{fin.} \psi (a \cos. \psi \pm \sqrt{}(b^2 - (d^2-a)^2 \text{fin.} \psi^2))}{b^2 - a^2 + (2a-d) d \text{fin.} \psi^2} - \varphi.$$

Coroll. IV.

§. 513. Si angulus A ſit rectus vel etiam aequalis tribus rectis, cadente vertice A ſupra diagonalem BD, et trapezio inverſo prodeunte, aequatio pro latere BC in hanc mutatur:

$$b = \frac{\sqrt{}(a^2 \text{fin.} (\beta+\varphi)^2 \cos. (\beta+\varphi)^2 + (d \text{fin.} \varphi \pm a \text{fin.} (\beta+\varphi)^2)^2)}{\text{fin.} (\beta+\varphi)}.$$

pro latere AB, $a = \pm d \text{fin.} \varphi \pm \sqrt{}(b^2 - d^2 \text{fin.} \varphi^2 \cot. (\beta+\varphi)^2)$, pro latere CD,

$d =$

$$d = \sin.(\beta+\phi)\frac{(\pm a \sin(\beta+\phi) \pm \surd(b^2 - a^2 \cos.(\beta+\phi)^2))}{\sin.\phi}.$$

Pro angulo ACD habetur haec aequatio:

$$tang.(\beta+\phi) = \frac{d \sin.\phi \surd(b^2 - (d \sin.\phi \mp a)^2)}{b^2 - a^2 \pm 2ad \sin.\phi - d^2 \sin.\phi^2},$$

vel etiam $tang.(\beta+\phi) = \frac{d \sin.\phi \surd(b^2 - (d \sin.\phi \mp a)^2)}{b^2 - (d \sin.\phi \pm a)^2},$

et angulus ipse

$$\beta = ang.\ tang.\frac{d \sin.\phi \surd(b^2 - (d \sin.\phi \mp a)^2)}{b^2(d \sin.\phi \mp a)^2} - \phi.$$

Coroll. V.

§. 514. Quod si angulus D ponatur rectus vel etiam aequalis tribus rectis, cadente D ad sinistram diagonalis et trapezio inverso prodeunte, aequatio pro latere BC in hanc mutatur:

$$b = \frac{\surd(a^2 \cos.\beta^2 \cos.(\beta+\psi)^2 + (d \mp a \cos.\beta \sin.(\beta+\psi))^2)}{\cos.\beta},$$

pro latere AB

$$a = \frac{\pm d \sin.(\beta+\psi) \pm \surd(b^2 \cos.\beta^2 - d^2 \cos.(\beta+\psi)^2)}{\cos.\beta},$$

pro latere CD, $d = \cos.\beta(\mp a \sin.(\beta+\phi) \pm \surd(b^2 - a^2 \cos.(\beta+\psi)))$, pro angulo A habetur:

$\psi = ang.\ \sin.\frac{(b^2 - a^2)\cos.\beta^2 - d^2}{\pm 2ad \cos.\beta} - \beta$, denique pro angulo ACD habetur:

$$\cot.\beta = \frac{d(\pm a \cos.\psi \pm \surd(b^2 - (a \sin.\psi \pm a)^2))}{b^2 - a^2 \pm 2ad \sin.\psi - d^2}$$

et angulus ipse

$$\beta = ang.\cot.\frac{d(\pm a \cos.\psi \pm \surd(b^2 - (a \sin.\psi \pm a)^2))}{b^2 - a^2 \pm 2ad \sin.\psi - d^2}.$$

Coroll.

Coroll. VI.

§. 515. Si angulus ACD ponatur rectus, aequatio pro latere BC in hanc abit:

$$b = \frac{\sqrt{(a^2 \cos.\varphi^2 \cos.(\varphi+\psi)^2 + (d \sin.\varphi \pm a \cos.\varphi \sin.(\varphi+\psi))^2)}}{\cos.\varphi}$$

vel etiam in hanc:

$$b = \sqrt{(a^2 \pm 2ad \sin.(\varphi+\psi) \tang.\varphi + d^2 \tang.\varphi^2)},$$

pro latere AB,

$$a = \frac{d \sin.\varphi \sin.(\varphi+\psi) \pm \sqrt{(b^2 \cos.\varphi^2 - d^2 \sin.\varphi^2 \cos.(\varphi+\psi)^2)}}{\cos.\varphi}$$

vel etiam

$$a = d \tang.\varphi \sin.(\varphi+\psi) \pm \sqrt{(b^2 - d^2 \cos.(\varphi+\psi)^2 \tang\varphi^2)},$$

pro latere DC habetur: $d = \cot.\varphi(\sin.(\varphi+\psi) \pm \sqrt{(b^2 - a^2 \cos.(\varphi+\psi)^2)})$, pro angulo A

$$\psi = ang.\sin.\frac{(a^2-b^2)\cos.\varphi^2 + d^2 \sin.\varphi^2}{2ad \sin.\varphi \cos.\varphi} - \varphi.$$

Coroll. VII.

§. 516. Si angulus A sit aequalis duobus rectis, lateribus AB et AD in diagonalem BD incidentibus, et trapezio in triangulum BCD abeunte pro latere BC prodit haec aequatio:

$$b = \frac{\sqrt{(a^2 \sin.(\beta+\varphi)^4 + (d \sin.\varphi \pm a \sin.(\beta+\varphi) \cos.(\beta+\varphi))^2)}}{\sin.(\beta+\varphi)},$$

pro latere AB, $a = d \sin.\varphi \cot.(\beta+\varphi) \pm \sqrt{(b^2 - d^2 \sin.\varphi^2)}$ pro latere CD

$$d = \frac{\sin.(\beta+\varphi)(a \cos.(\beta+\varphi) \pm \sqrt{(b^2 - a^2 \sin.(\beta+\varphi)^2)})}{\sin.\varphi}$$

pro

pro angulo ACD

$$tang.(\beta+\varphi)=\frac{d\,fin.\,\varphi\,(-a\pm\sqrt{}\,(b^2-d^2\,fin.\,\varphi^2))}{b^2-a^2-d^2\,fin.\,\varphi^2}$$

et angulus ipſe,

$$\beta=ang.\,tang.\frac{d\,fin.\,\varphi\,(-a\pm\sqrt{}\,(b^2-d^2\,fin.\,\varphi^2))}{b^2-a^2-d^2\,fin.\,\varphi^2}-\varphi.$$

Scholion II.

§. 517. Per hanc poſitionem in ultimo corollario adhibitam ſolvitur unum ad triangula ſpectans problema. puta, quando quaeritur latus DC; nam in triangulis partialibus non dantur niſi duo, nec in totali habentur niſi duo, cum ſegmento BE lateris BD, et angulo partiali ECD; alterum etiam ad triangula ſpectans problema ſolveretur, quando quaeritur angulus D, ſiquidem aequatio ſexti gradus, quam non evolvi, ſolutionem ſatis commodam utilemque praeberet. Caeteri caſus etiamſi nova quadam methodo ſolvantur, tamen etiam trigonometrice expediuntur, caſus prior etiam aliis methodis, hic poſterior non ſolvitur.

Coroll VIII.

§. 518. Duo dantur caſus in quibus tangens anguli ADC ab irrationalitate liberatur, quorum unus obtinet, ſi fuerit $d\,fin.\,\varphi=a\,fin.\,\psi$ et conſequenter $d:fin.\,\psi=a:fin.\,\varphi$, h. e. quando latus AB, DC, ſunt inter ſe, ut ſinus angulorum ad baſin AD ipſis oppoſitorum, in hoc enim caſu fit: $tang.(\beta+\varphi)=\frac{d\,fin\,\varphi\,(a\,cos.\,\psi\pm b)}{b^2-a^2+d^2\,fin.\,\varphi^2}$,

et

et angulus ipſe, $\beta = ang.tang. \frac{d\, fin.\varphi\,(a\, cos.\psi \pm b)}{b^2 - a^2 + d^2 fin.\varphi^2} - \varphi$, alter caſus obtinet quando eſt, $b = d\, fin.\varphi - a\, fin.\psi$ in quo fit $tang.(\beta+\varphi) = \frac{-d\, fin.\varphi}{a}$ et ipſe angulus

$$\beta = ang.tang. \frac{-d\, fin.\varphi}{a} - \varphi.$$

Scholion III.

§. 519. Si latus AD cadat intra latus AB, quomodocunque trapezium fit inverſum, valet haec aequatio: $(b^2 - a^2)\, fin.(\beta+\varphi)^2 = d^2 fin.\varphi^2 \pm 2\, a\, d\, fin.\varphi\, fin.(\beta+\varphi)\, cos.(\beta+\varphi+\psi)$. Si vero latus AB fit intra latus AD, quomodocunque trapezium fit inverſum, valet aequatio primo inventa, proinde generalis aequatio haec eſt: $(b^2 - a^2)\, fin.(\beta+\varphi)^2 = d^2 fin.\varphi^2 \pm 2\, a\, d\, fin.\varphi\, fin.(\beta+\varphi)\, cos.(\beta+\varphi \pm \psi)$.

Problema XXXVII.

§. 520. In figura quadrilatera propoſita $ABCD$, inter haec ſex: latus $AB = a$, $DC = d$, angulum $A = \psi$, $D = \varphi$, $B = \lambda$, $ACD = \beta$, aequationem invenire. Fig. XXXVII.

Solutio.

Cum ex antecedentibus conſtet eſſe $AC = \frac{d\, fin.\varphi}{fin.(\beta+\varphi)}$ et $fin.ACB = -fin.(\beta+\lambda+\varphi+\psi)$; in triangulo ſiniſtro pervenio ſtatim ad hanc analogiam: $AB : fin.ACB = AC : fin.B$, et ſymbolis

bolis substitutis $a: -\ \textit{fin.}\ (\beta+\lambda+\phi+\psi) = \frac{d\ \textit{fin.}\ \phi}{\textit{fin.}\ (\beta+\phi)} : \textit{fin.}\ \lambda$, consequenter etiam habetur haec aequatio: $a\ \textit{fin.}\ \lambda\ \textit{fin.}\ (\beta+\phi) = -\ d\ \textit{fin.}\ \phi\ \textit{fin.}\ (\beta+\lambda+\phi+\psi)$. Q. E. I.

Coroll. I.

§. 521. Ex hac aequatione statim habentur latera cum sit: $a = \frac{-d\ \textit{fin.}\ \phi\ \textit{fin.}\ (\beta+\lambda+\phi+\psi)}{\textit{fin.}\ \lambda\ \textit{fin.}\ (\beta+\phi)}$, nec non $d = \frac{-a\ \textit{fin.}\ \lambda\ \textit{fin.}\ (\beta+\phi)}{\textit{fin.}\ \phi\ \textit{fin.}\ (\beta+\lambda+\phi+\psi)}$, faciliter etiam habetur angulus A, cum sit, $\textit{fin.}\ (\beta+\lambda+\phi+\psi) = \frac{-a\ \textit{fin.}\ \lambda\ \textit{fin.}\ (\beta+\phi)}{d\ \textit{fin.}\ \phi}$ et consequenter,

$$\psi = \textit{ang.fin.} \frac{-a\ \textit{fin.}\ \lambda\ \textit{fin.}\ (\beta+\phi)}{d\ \textit{fin.}\ \phi} - \beta - \lambda - \phi.$$

Alia solutio obtinetur, sinus a se invicem separando, nam posito ad abbreviandum: $\textit{fin.}\ \lambda = h$, $\textit{fin.}\ \phi = k$, $\textit{fin.}\ (\beta+\phi) = m$, $\textit{fin.}\ (\beta+\lambda+\phi) = p$, $\textit{cos.}\ (\beta+\lambda+\phi) = q$, $\textit{fin.}\ \psi = x$, $\textit{cos.}\ \psi = \sqrt{(1-x^2)}$, et facta transpositione et substitutione prodit haec aequatio abbreviata: $ahm + dkqx = -\ dkp \sqrt{(1-x^2)}$, ex qua operatione absoluta elicitur haec aequatio:

$$x = \frac{-\ amhq \pm p \sqrt{(d^2k^2 - a^2h^2m^2)}}{dk} = \textit{fin.}\ \psi.$$

Coroll.

Coroll. II.

§. 522. Pro angulo B obtinendo habetur haec aequatio:

$$tang.\,\lambda = \frac{-d\,fin.\,\varphi\,fin.(\beta+\varphi+\psi)}{a\,fin.(\beta+\varphi)+d\,fin.\,\varphi\,cos.(\beta+\varphi+\psi)}$$

vel etiam haec:

$$tang.\,\lambda = \frac{-d\,fin.\,\varphi\,tang.\,(\beta+\varphi+\psi)}{d\,fin.\,\varphi + a\,fin.(\beta+\varphi)\,fec.\,(\beta+\varphi+\psi)},$$

fimiliter pro angulo ACD habetur haec:

$$tang.\,(\beta+\varphi) = \frac{-d\,fin.\,\varphi\,fin.\,(\lambda+\psi)}{a\,fin.\,\lambda + d\,fin.\,\varphi\,cos.\,(\lambda+\psi)}$$

$$= \frac{-d\,fin.\,\varphi\,tang.\,(\lambda+\psi)}{d\,fin.\,\varphi + a\,fin.\,\lambda\,fec.\,(\lambda+\psi)}$$ et hinc ad angulum defcendendo obtinetur:

$$\beta = ang.tang.\frac{-d\,fin.\,\varphi\,tang.\,(\lambda+\psi)}{d\,fin.\,\varphi + a\,fin.\,\lambda\,fec.\,(\lambda+\psi)} - \varphi.$$

Coroll. III.

§. 523. Ad obtinendum angulum D, feparatis a fe invicem finibus et pofito ad abbreviandum $fin.\,\lambda = h$, $fin.\,\beta = m$, $cos.\,\beta = n$, $fin.\,(\beta+\lambda+\psi) = p$, $cos.\,(\beta+\lambda+\psi) = q$, $fin.\,\varphi = x$, $cos.\,\varphi = \surd(1-x^2)$, facta fubftitutione et transpofitione habetur haec aequatio abbreviata: $ahnx + dqx^2 = -(ahm + dpx)\surd(1-x^2)$, ex qua operatione abfoluta, propter $nq + mp\,cos.(\lambda+\psi) = k$, brevitatis gratia, habetur haec aequatio quarti gradus: $x^4 + \frac{2ahkx^3}{d} + \frac{(a^2h^2-p^2)}{d^2}x^2 - \frac{2ahmpx}{d} - \frac{a^2h^2m^2}{d^2} = 0.$

Schol.

Scholion I.

§. 524. Inter fex cafus in aequatione contentos, duo dantur Tetragonometriae proprii, qui tum obtinent, cum in triangulo dextro quaeruntur anguli D et ACD, et in Geometria practica conftituunt duo problemata nova utilia juxta ac pulchra. Caeteri quatuor cafus utrique methodo et tetragonometricae et trigonometricae fubfunt, quos tamen evolvi, cum ex aequatione facile confequantur, et folutiones tetragonometricae trigonometricis non adeo multum cedunt.

Coroll. IV.

§. 525. Si anguli A et D ponantur fimul fumti aequales duobus rectis, ut prodeat trapezium parallelarum bafium AB, DC, aequatio pro latere AB in hanc mutatur:

$a = \frac{d \, fin. \varphi \, fin. (\beta + \lambda)}{fin. \lambda \, fin. (\beta + \varphi)}$ vel etiam angulo φ eliminato: $a = \frac{d \, fin. \psi \, fin. (\beta + \lambda)}{fin. \lambda \, fin. (\beta - \psi)}$, et pro latere DC erit, $d = \frac{a \, fin. \lambda \, fin. (\beta + \varphi)}{fin. \varphi \, fin. (\beta + \lambda)} = \frac{a \, fin. \lambda \, fin. (\beta - \psi)}{fin. \psi \, fin. (\beta + \lambda)}$, pro angulo B habetur:

$$tang. \lambda = \frac{d \; fin. \; \varphi \; fin. \; \beta}{a \; fin. \; (\beta + \varphi) - d \; fin. \; \varphi \; cos. \; \beta}$$

$$= \frac{d \; fin. \; \psi \; fin. \; \beta}{a \; fin. \; (\beta - \psi) - d \; fin. \; \psi \; cos. \; \beta} \text{ vel}$$

$$tang. \lambda = \frac{d \; tang. \; \beta}{a \; fin. \; (\beta + \varphi) \; fec. \; \beta \; cofec. \; \varphi - d}$$

$$= \frac{d \; tang. \; \beta}{a \; fin. \; (\beta - \psi) \; fec. \; \beta^2 \; cofec. \; \varphi - d}$$

pro

pro angulo ACD

$$tang.(\beta+\varphi)=\frac{-d\,sin.\psi\;sin.(\lambda+\psi)}{a\;sin.\;\lambda+d\;sin.\;\psi\;cos.\;(\lambda+\psi)}$$

vel etiam,

$$tang.\;(\beta+\varphi)=\frac{-d\;tang.\;(\lambda+\psi)}{a\,sin.\;\lambda\;sec.\;(\lambda+\psi)\;cosec.\;\psi+d}.$$

Coroll. V.

§. 526. Si anguli A et B ſimul ſumti ſint aequales duobus rectis, prodeunte trapezio parallelarum baſium, AD, BC aequatio pro latere AB in hanc mutatur: $a=\frac{d\,sin.\;\varphi}{sin.\;\lambda}=\frac{d\;sin.\;\varphi}{sin.\;\psi}$, ex qua aequatione ſimpliciſſima caeteri caſus ſtatim perſpiciuntur. Denique ſi anguli diagonaliter oppoſiti B et D ſint ſimul ſumti aequales duobus rectis, ut prodeat trapezium circulo inſcriptibile, aequatio pro latere AB in hanc mutatur: $a=\frac{d\,sin.\;\varphi\,sin.(\beta+\psi)}{sin.\;\lambda\;sin.(\beta+\varphi)}=\frac{d\;sin.\;(\beta+\psi)}{sin.\;(\beta+\varphi)}$, et hinc $d=\frac{a\,sin.(\beta+\varphi)}{sin.\;(\beta+\psi)}$ et pro angulo A habetur haec aequatio angularis:

$\psi=ang.sin.\frac{a\;sin.\;(\beta+\varphi)}{d}-\beta$, ſed in aequatione quadratica fit, $h=k$, $p=-sin.\beta$, $q=-cos.\beta$, quae igitur in hanc mutatur:

$x=\frac{amq\mp p\sqrt{(d^2-a^2m^2)}}{d}=sin.\;\psi$, et pro an-

Z gulo

gulo ACD, $\text{tang.}\,(\beta+\varphi) = \frac{-d\ \text{ſin.}\,(\lambda+\psi)}{a+d\ \text{cos.}\,(\lambda+\psi)}$ $= \frac{-d\ \text{tang.}\,(\lambda+\psi)}{d+a\ \text{ſec.}\,(\lambda+\psi)}$ et hinc

$$\beta = \text{ang. tang.} \frac{-d\ \text{tang.}\,(\lambda+\psi)}{d+a\ \text{ſec.}\,(\lambda+\psi)} - \varphi.$$

Coroll. VI.

§. 527. Si angulus A ſit rectus vel etiam tribus rectis aequalis, cadente vertice A ſupra diagonalem BD, et trapezio inverſo prodeunte, aequatio pro latere AB in hanc mutatur:

$a = \frac{\mp d\ \text{ſin.}\,\varphi\ \text{cos.}\,(\beta+\lambda+\varphi)}{\text{ſin.}\,\lambda\ \text{ſin.}\,(\beta+\varphi)}$, pro latere DC,

$d = \frac{\mp a\,\text{ſin.}\,\lambda\ \text{ſin.}\,(\beta+\varphi)}{d\ \text{ſin.}\,\varphi\ \ \text{cos.}\,(\beta+\lambda+\varphi)}$, pro angulo B

$\text{tang.}\,\lambda = \frac{\mp d\ \text{ſin.}\,\varphi\ \text{cos.}\,(\beta+\varphi)}{a\ \text{ſin.}\,(\beta+\varphi) \mp d\ \text{ſin.}\,\varphi\ \text{ſin.}\,(\beta+\varphi)}$

$= \frac{d\ \text{ſin.}\,\varphi}{(d\ \text{ſin.}\,\varphi \pm a)\ \text{tang.}\,(\beta+\varphi)}$ pro angulo ACD,

$\text{tang.}\,(\beta+\varphi) = \frac{\mp d\ \text{ſin.}\,\varphi\ \text{cos.}\,\lambda}{a\,\text{ſin.}\,\lambda \mp d\ \text{ſin.}\,\varphi\ \text{ſin.}\,\lambda}$ vel etiam

$\text{tang.}\,(\beta+\psi) = \frac{d\ \text{ſin}\ \varphi}{(d\ \text{ſin.}\,\varphi \pm a)\ \text{tang.}\,\lambda}$, et angulus

ipſe $\beta = \text{ang. tang.} \frac{d\ \text{ſin.}\,\varphi}{(d\ \text{ſin.}\,\varphi \pm a)\ \text{tang.}\,\lambda} - \varphi.$

In aequatione quarti gradus pro angulo D fit, $p = \pm\,\text{cos.}\,(\lambda+\beta)$, $q = \mp\,\text{ſin.}\,(\beta+\lambda)$, $k = \mp\,\text{ſin.}\,\lambda$, quare aequatio in hanc paululum mutatur:

$$x^4 \mp \frac{2ahkx^3}{d} + \frac{(a^2h^2-p^2)\,x^2}{d^2} \mp \frac{2ahmpx}{d} - \frac{a^2h^2m^2}{d^2} = 0.$$

Coroll.

Coroll. VII.

§. 528. Si angulus D ſit aequalis recto vel tribus rectis, cadente vertice D ad ſiniſtram diagonalis, et trapezio inverſo prodeunte, pro latere $A\,B$ aequatio in hanc mutatur:
$a = \frac{\mp\, d\, cos.\,(\beta+\lambda+\psi)}{ſin.\,\lambda\, cos.\,\beta}$ et pro latere DC,
$d = \frac{\mp\, a\, ſin.\,\lambda\, cos.\,\beta}{cos.\,(\beta+\lambda+\psi)}$, pro angulo A aequatio angularis: $\psi = ang.\, cos. \frac{\mp\, a\, ſin.\,\lambda\, cos.\,\beta}{d} - \beta - \lambda$,
ſed in aequatione quadratica fit, $k = \pm\, 1$, $m = \pm\, cos.\,\beta$, $p = \pm\, cos.\,(\lambda+\beta)$, $q = \mp\, ſin.\,(\beta+d)$, quae igitur in hanc paululum mutatur:
$x = \frac{\pm\, a\, m\, k\, q \pm p\, \surd(d^2 - a^2 h^2 m^2)}{d} = ſin.\ \psi$;
pro angulo B, $tang.\,\lambda = \frac{-\,d\, cos.\,(\beta+\psi)}{\pm\, a\, cos.\,\beta - d\, ſin.\,(\beta+\psi)}$
$= \frac{d}{d\, tang.\,(\beta+\psi) \mp a\, cos.\,\beta\, ſec.\,(\beta+\phi)}$ pro angulo ACD, $cot.\,\beta = \frac{d\ tang.\,(\lambda+\psi)}{d \pm a\, ſin.\,\lambda\, ſec.\,(\lambda+\psi)}$ vel
$tang.\,\beta = \frac{d \pm a\, ſin.\,\lambda\, ſec.\,(\lambda+\psi)}{d\ tang.\,(\lambda+\psi)}$, et angulus ipſe,
$\beta = ang.\ tang. \frac{d \pm a\, ſin.\,\lambda\, ſec.\,(\lambda+\psi)}{d\ tang.\,(\lambda+\psi)}$.

Coroll. VIII.

§. 529. Si angulus B ſit rectus vel etiam tribus rectis aequalis, cadente vertice B ad

 dex-

dextram diagonalis AC, et trapezio inverso prodeunte fit pro latere AB aequatio:

$$a = \frac{-d\, fin.\, \varphi\, cos.\, (\beta+\varphi+\psi)}{fin.\, (\beta+\varphi)},$$ et pro latere DC

$$d = \frac{-a\, fin.\, (\beta+\varphi)}{fin.\, \varphi\, cos.\, (\beta+\varphi+\psi)},$$ pro angulo A aequatio angularis: $\psi = ang.\, cos. \frac{-a\, fin.\, (\beta+\varphi)}{d\, fin.\, \varphi} - \beta - \varphi$.

Sed in aequatione quadratica fit, $h = \pm 1$, $p = \pm cos.\, (\beta+\varphi)$, $q = \mp fin.\, (\beta+\varphi)$, aequatio igitur in hanc paululum mutatur:

$$x = \frac{a\, m\, q \pm p \sqrt{(d^2 k^2 - a^2 m^2)}}{d\, k} = fin.\, \psi,$$

pro angulo ACD: $tang.(\beta+\varphi) = \frac{d\, fin.\, \varphi\, cos.\, \psi}{d\, fin.\, \varphi\, fin.\, \psi - a}$

$$= \frac{d\, fin.\, \varphi}{d\, fin.\, \varphi\, tang.\, \psi - a\, fec.\, \psi},$$ et angulus ipse,

$$\beta = ang.\, tang. \frac{d\, fin.\, \varphi}{d\, fin\, \varphi\, tang.\, \psi \mp a\, fec.\, \psi} - \varphi.$$ Denique in aequatione quarti gradus fit, $h = \pm 1$, $p = \pm cos.(\beta+\psi)$, $q = \pm fin.(\beta+\psi)$, et $k = \mp fin.\, \psi$, quare aequatio in hanc paululum mutatur:

$$x^4 - \frac{2ak x^3}{d} + \frac{(a^2 - p^2) x^2}{d} - \frac{2ampx}{d} - \frac{a^2 m^2}{} = 0.$$

Coroll. IX.

§. 530. Si angulus A ponatur aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus et trapezio in triangulum BCD abeunte; aequatio pro segmento BE lateris BD in hanc mutatur:

$a =$

$a = \frac{d\, fin.\, \phi\, fin.\, (\beta+\lambda+\phi)}{fin.\, \lambda\, fin.\, (\beta+\phi)}$, et pro latere DC in hanc: $d = \frac{a\, fin.\, \lambda\, fin.\, (\beta+\phi)}{d\, fin.\, \phi\, fin.\, (\beta+d+\phi)}$ et pro angulo B

$tang.\, \lambda = \frac{d\, fin.\, \phi\, tang.\, (\beta+\phi)}{a\, tang.\, (\beta+\phi) - d\, fin.\, \phi}$

$= \frac{d\, tang.\, (\beta+\phi)}{a\, tang.\, (\beta+\phi)\, cofec.\, \phi - d}$. Pro angulo ACD: $tang.\, (\beta+\phi) = \frac{d\, fin.\, \phi\, tang.\, \lambda}{a\, tang.\, \lambda - d\, fin.\, \phi}$, vel etiam: $tang.\, (\beta+\phi) = \frac{d\, tang.\, \lambda}{a\, tang.\, \lambda\, cofec.\, \phi - d}$ et angulus $\beta = ang.\, tang.\, \frac{d\, tang.\, \lambda}{a\, tang.\, \lambda\, cofec\, \phi - d} - \phi$.

Sed in aequatione quarti gradus pro angulo D fit, $p = -fin.\, (\beta+\lambda)$, $q = -cos.\, (\beta+\lambda)$ et $k = -cos.\, \lambda$, quare aequatio in hanc paululum mutatur: $x^4 - \frac{2ahkx^3}{d} + \frac{(a^2h^2 - p^2)}{d^2}$

$x^2 + \frac{2ahmpx}{d} - \frac{a^2h^2m^2}{d^2} = 0.$

Scholion II.

§. 531. Hac pofitione ultimi Corollarii unum novum ad triangula fpectans problema folvitur, quod tum obtinet, quando quaeritur angulus D, cum in triangulo totali non dentur nifi duo, fegmentum BE et angulis B, et partialis ECD, nec in partialibus nifi duo, quod problema vix ullis aliis methodis folvitur; caeteri

 cafus

casus etiam si nova quadam methodo tetragonometrica solvantur, tamen etiam trigonometrice expediuntur.

Scholion III.

§. 532. Si latus AD cadat intra latus AB, quomodocunque trapezium sit inversum; obtinet haec aequatio: $a\, \mathit{fin.}\, \lambda\, \mathit{fin.}\, (\beta + \phi) = d\, \mathit{fin.}\, \phi\, \mathit{fin.}\, (\beta + \phi - \psi - \lambda)$, si vero latus AB cadat intra latus AD, quomodocunque trapezium sit inversum, obtinet haec aequatio: $a\, \mathit{fin.}\, \lambda\, \mathit{fin.}\, (\beta + \phi) = d\, \mathit{fin.}\, \phi\, \mathit{fin.}\, (\beta + \phi + \psi - \lambda)$, est igitur aequatio universalis hujus formae: $a\, \mathit{fin.}\, \lambda\, \mathit{fin.}\, (\beta + \phi) = \mp\, d\, \mathit{fin.}\, \phi\, \mathit{fin.}\, (\beta + \phi \pm \psi \pm \lambda)$.



CAPUT

CAPUT XI.

Continens tria problemata quintae claſſis particularis, ſub poſteriore principali contentae.

Problema XXXVIII.

§. 533. In figura quadrilatera propoſita $ABCD$, inter haec ſex: latus $BC=b$, $CD=d$, $AD=c$, angulum $A=\psi$, $B=\lambda$, $ACD=\beta$, aequationem invenire Fig. XXXVIII.

Solutio.

Cum per aetecedentia probl. 30. (§. 427.) conſtet eſſe diagonalem: $AC = d\ cos.\ \beta + \surd(c^2 - d^2\ ſin.\ \beta^2)$ atque

$$ſin.\ CAB = \frac{ſin.\ \psi \surd (c^2 - d^2\ ſin.\ \beta^2) - d\ ſin.\ \beta\ cos\ \beta}{c},$$

in triangulo ſiniſtro pervenio ſtatim ad hanc analogiam AC; $ſin.\ B = BC : ſin.\ CAB$, h. e. ſymbolis ſubſtitutis: $d\ cos.\ \beta + \surd(c^2 - d^2\ ſin.\ \beta^2)$:

$$ſin.\ \lambda = b : \frac{ſin.\ \psi \surd (c^2 - d^2\ ſin.\ \beta^2) - \lambda\ ſin.\ \beta\ cos.\ \psi}{c}$$

adeoque habetur haec aequatio: $b\ c\ ſin.\ \lambda = d\ ſin.\ (\psi - \beta) \surd (c^2 - d^2\ ſin\ \beta^2) - d^2\ ſin.\ \beta\ cos.\ (\psi - \beta) + c^2\ ſin.\ \psi$. Q. E. I.

Coroll. I.

§. 534. Ex hac aequatione facillime ſolvuntur duo caſus, quorum unus, ſi inveniendum

 latus

latus BC, cum sit: $b = \frac{d\,\text{fin.}(\psi-\beta)\surd(c^2-d^2\,\text{fin.}\,\beta^2)}{c\,\text{fin.}\,\lambda}$ $\frac{-d^2\,\text{fin.}\,\beta\,\text{cos.}(\psi-\beta)+c^2\,\text{fin.}\,\psi}{c\,\text{fin.}\,\lambda}$, alter si inveniendus angulus B, cum sit: $\text{fin.}\,\lambda = \frac{d\,\text{fin.}(\psi-\beta)}{bc}$ $\frac{\surd(c^2-d^2\,\text{fin.}\,\beta^2)-d^2\,\text{fin.}\,\beta\,\text{cos.}(\psi-\beta)+c^2\,\text{fin.}\,\psi}{bc}$

Coroll. II.

§. 535. Ad inveniendum latus CD, sit ad abbreviandum $\text{fin.}\,\lambda = h$, $\text{fin.}\,\psi = m$, $\text{fin.}\,\beta = n$, $\text{fin.}(\psi-\beta) = p$, $\text{cos.}(\psi-\beta) = q$, et facta substitutione et transpositione habetur: $c(bh-cm)+d^2nq = dp\surd(c^2-d^2n^2)$, ex qua operatione ad finem perducta obtinetur: $d = \frac{\surd(c^2p^2-2cnq(bh-cm)\pm cp}{n\surd 2}$ $\frac{\surd(c^2p^2-4n(bh-cm)(cq+bnh-cmn)))}{n\surd 2}$, ex eadem aequatione abbreviata pro latere AD obtinendo pervenitur ad hanc aequationem quarti gradus: $c^4 - \frac{2bh}{m}c^3 + (\frac{b^2h^2}{m^2} - \frac{d^2p^2}{m^2} - \frac{2d^2nq}{m})$ $c^2 + \frac{2d^2bnhqc}{m^2} + \frac{d^4n^2}{} = 0.$

Coroll. III.

§. 536. Ad inveniendum angulum A, pono ad abbreviandum $\text{fin.}\,\lambda = h$, $\text{fin.}\,\beta = m$, $\text{cos.}\,\beta = n$, $\text{fin.}\,\psi = x$. $\text{cos.}\,\psi = \surd(1-x^2)$, $\surd(c^2-d^2m^2) = r$; facta

facta ſeparatione ſinuum, ſubſtitutione et transpoſitione, habetur haec aequatio abbreviata: $bch-(c^2+(rn-dm)\,d)\,x = -dm\,(r+dn)\,\surd(1-x^2)$; ſitque rurſus ad abbreviandum coefficiens ipſius, $x = P$, et coefficiens ipſius $\surd(1-x^2) = Q$, et facta ſubſtitutione, erit aequatio decurtata: $bch - Px = -Q\surd(1-x^2)$, ex qua operatione abſoluta obtinetur:

$$x = \frac{Pbch \pm Q\surd(P^2+Q^2-b^2c^2h^2)}{P^2+Q^2} = \textit{ſin.}\ \psi.$$

Ad inveniendum angulum ACD, ſit ad abbreviandum, $bc\ \textit{ſin.}\ \lambda - c^2\ \textit{ſin.}\ \psi = A^2$, et facta ſubſtitutione erit aequatio abbreviata: $A^2 + d^2\ \textit{ſin.}\ \beta\ \textit{cos.}\ (\psi-\beta) = d\ \textit{ſin.}\ (\psi-\beta)\ \surd(c^2-d^2\ \textit{ſin.}\ \beta^2)$, et ſumtis quadratis, facta concinnatione, et termino A^4 per $\textit{ſin.}\ \beta^2 + \textit{cos.}\ \beta^2$ multiplicato, ad aequationem reſpectu anguli β homogeneam reddendam, et facta transpoſitione et diviſione, et poſito ad abbreviandum $\textit{ſin.}\ \psi = m$, $\textit{cos.}\ \psi = n$, $\textit{taug.}\ \beta = t$, proveniet haec aequatio: $A^4 + 2A^2 d^2 m + d^4 - d^2c^2n^2)\ t^2 + 2d^2nt\,(A^2+c^2m) = d^2c^2m^2 - A^4$ ex qua operatione abſoluta obtinetur:

$$t = \frac{-d^2n\,(A^2+c^2m) \pm \surd\big(d^2n^2(A^2+c^2m)^2}{A^4+2A^2d^2m+d^4-d^2c^2n^2}$$
$$\frac{+(d^2c^2m^2-A^4)\ (A^4+2A^2d^2m+d^4-d^2c^2n^2)\big)}{A^4+2A^2d^2m+d^4-d^2c^2n^2}$$

Scholion I.

§. 537. Inter ſex caſus qui in aequatione continentur, ſunt tantum tres Tetragonometriae proprii, qui tum obtinent, cum ex tribus illis, quae in triangulo dextro ſignata ſunt, ſigillatim

Z 5 ſumtis

ſumtis et inveniendis, quaeritur conſtructio figurae, et in Geometria practica totidem problemata nova ſatis pulchra et utilia conſtituunt. Caeteri tres caſus utrique methodo et tetragonometricae et trigonometricae ſubſunt, ſed ita ut trigonometrice fere facilius expediantur, quorum tamen ſolutiones tetragonometricas dedi, quia ex aequatione facile conſequuntur.

Coroll. IV.

§. 538. Si anguli A et B ponantur ſimul ſumti aequales duobus rectis, prodeunte ſic trapezio parallelarum baſium BC, AD, aequatio pro latere BC in hanc mutatur, angulo λ eliminato:

$$b = \frac{d\,\mathit{ſin.}(\psi - \beta)\sqrt{(c^2 - d^2\,\mathit{ſin.}\,\beta^2)} + d^2\,\mathit{ſin.}\,\beta\,\cos.(\psi - \beta) + c}{c\,\mathit{ſin.}\,\psi},$$

at angulo ψ eliminato habetur:

$$b = \frac{d\,\mathit{ſin.}(\beta + \lambda)\sqrt{(c^2 - d^2\,\mathit{ſin}\,\beta^2)} + d^2\,\mathit{ſin.}\,\beta\,\cos.(\beta + \lambda) + c^2\,\mathit{ſin.}\,\lambda}{c\,\mathit{ſin}\,\lambda};$$

in aequatione quadratica pro latere CD fit $h = m$, et aequatio in hanc paululum mutatur:

$$d = \frac{\sqrt{(c^2p^2 - 2cmnq(b - c) \pm cp\sqrt{(c^2p^2 - 4mn(b - c)(cq + (b - c)mn))})}}{n\sqrt{2}}.$$

Sed aequatio quarti gradus pro latere AD in hanc etiam paululum mutatur:

$$c^4 - 2bc^3 + \left(b^2 - \frac{d^2p^2}{m^2} - \frac{d^2nq}{m}\right)c^2 + \frac{2d^2bnqc}{m} + \frac{d^4n^2}{m^2} = 0.$$

In

In aequatione quadratica pro angulo ACD, fit, $A^2=(b-c)\ c\ fin.\ \lambda=(b-c)\ c\ fin.\ \psi$, $m=fin.\ \lambda=fin.\ \psi$, et $n=cos.\ \psi=-cos.\ \lambda$, fed aequatio quoad formam externam invariata manet.

Coroll. V.

§. 539. Si angulus A fit rectus vel etiam aequalis duobus rectis, cadente vertice A fupra diagonalem BD, et trapezio inverfo prodeunte, aequatio pro latere BC in hanc mutatur:

$$b=\frac{d\ cos.\ \beta\ \sqrt{(c^2-d^2 fin.\ \beta^2)}\pm d^2 fin.\ \beta^2\pm c^2}{c\ fin.\ \lambda},$$

pro angulo β fit,

$$fin.\ \lambda=\frac{d\ cos.\ \beta\ \sqrt{(c^2-d^2 fin.\ \beta^2)}\pm d^2 fin.\ \beta^2\pm c^2}{b};$$

in aequatione quadratica pro latere CD fit, $m=\pm 1$, $p=\pm cos.\ \beta$, $q=\mp n$, quae igitur in hanc mutatur: $d=\frac{\sqrt{(c^2p^2-2cn^2(c\pm bh)}\pm cp}{n\sqrt{2}}$

$$\frac{\sqrt{(c^2p^2-4n(bh\mp c)(bh\mp 2c)n))}}{n\sqrt{2}};$$ fed aequatio quarti gradus pro latere AD in hanc mutatur: $c^4\mp 2bhc^3+(b^2h^2-d^2p^2+2d^2n^2)\ c-2d^2bhn^2+d^4n^2=0$; aequatio vero pro angulo ACD in hanc mutatur:

$$tang.\ \beta=\frac{\sqrt{(d^2c^2-A^4)}}{A^2\pm d^2}.$$

Coroll. VI.

§. 540. Si angulus B fit rectus vel etiam aequalis tribus rectis, cadente vertice B ad dextram

tram diagonalis AC, et trapezio inverſo prodeunte, aequatio pro latere BC in hanc mutatur:

$$b = \frac{d\, \text{ſin.}\, (\psi - \beta)\, \sqrt{(c^2 - d^2\, \text{ſin.}\, \beta \mp d^2\, \text{ſin.}\, \beta}}{c}$$

$$\frac{\cos.\, (\psi - \beta) \pm c^2\, \text{ſin.}\, \psi}{c},$$ in aequatione quadratica pro latere DC fit, $h = \pm 1$, quae igitur in hanc paululum mutatur: $$d = \frac{\sqrt{(c^2 p^2 + 2cnq\, (cm \mp b)}}{n\sqrt{2}}$$

$$\frac{\pm cp\sqrt{(c^2 p^2 + 4n(cm \mp b)\, (cq \pm bn - cmn))})}{n\sqrt{2}},$$

ſed aequatio quarti gradus pro latere AD in hanc abit: $$c^4 \mp \frac{2b}{m} c^3 + (\frac{b^2}{m^2} - \frac{d^2 p^2}{m^2} - \frac{2d^2 nq}{m})\, c^2 \pm \frac{2d^2 bnqc}{m^2} + \frac{d^4 n^2}{m^2} = 0.$$ In aequatione corollarii tertii pro angulo A fit etiam $h = \pm 1$, quae igitur in hanc paululum mutatur:

$$x = \pm \frac{Pbc \pm Q\sqrt{(C^2 + Q^2 - b^2 c^2)}}{P^2 + Q^2} = \text{ſin.}\ \psi.$$

In aequatione ejusdem corollarii pro angulo ACD, fit; $A = \pm bc - c^2\, \text{ſin.}\, \psi$, ſed ipſa aequatio quoad formam externam invariata manet.

Coroll. VII.

§. 541. Si denique angulus ACD ſit rectus vel etiam aequalis tribus rectis, cadente latere DC ſupra latus CB, et vertice D ad ſiniſtram diagonalis AC, et trapezio inverſo prodeunte, fit latus $$BC = b = \frac{\cos.\, \psi \sqrt{(c^2 - d^2) + (c^2 - d^2)\, \text{ſin.}\, \psi}}{c\, \text{ſin.}\, \lambda}$$

et

et hinc pro angulo B

$$\text{fin.}\ \lambda = \frac{\cos.\ \psi \sqrt{(c^2-d^2)} + (c^2-d^2)\ \text{fin.}\ \psi}{c};$$ in

aequationibus fecundi corollarii fit, $n = \pm 1$, $p = \mp \cos.\ \psi$, $q = \pm \text{fin.}\ \psi$, quare prior aequatio pro latere CD in hanc mutatur:

$$d = \frac{\sqrt{(c^2 p^2 - 2cq(bh - cm)} \pm cp}{\sqrt{(2}}$$

$$\frac{\sqrt{(c^2p^2 - 4(bh - cm)(cq \pm bh \mp cm)))}}{\sqrt{(2}}$$

et pofterior quarti gradus in hanc abit:

$$c^4 - \frac{2bhc^3}{m} + \left(\frac{b^2h^2}{m^2} - \frac{d^2p^2}{m^2} - \frac{2d^2q}{m}\right)c^2 + \frac{2d^2bhqc}{m^2} + \frac{d^4}{m^2} = 0.$$

In aequatione corollarii III. pro angulo A, fit $m = \pm 1$, $n = 0$, $r = \sqrt{(c^2 - d^2)}$, $P = bch - c^2 \mp d^2$, $Q = -dmr$, verum quoad formam externam aequatio invariata manet.

Coroll. VIII.

§. 542. Si angulus A ponatur duobus rectis aequalis, lateribus AB, AD in diagonalem BD incidentibus et trapezio in triangulum BCD abeunte, fit latus BC,

$$b = \frac{d\ \text{fin.}\ \beta\ (\sqrt{(c^2 - d^2\ \text{fin.}\ \beta^2)} + d\ \cos.\ \beta)}{c\ \text{fin.}\ \lambda}$$ et

$$\text{fin.}\ \lambda = \frac{d\ \text{fin.}\ \beta(\sqrt{(c^2 - d^2\ \text{fin.}\ \beta^2)} + d\ \cos.\ \beta)}{bc}.$$

In aequatione quadratica pro latere CD, et quarti gradus pro latere AD, fit $m = 0$, $p = \text{fin.}\ \beta = n$, $q = -\cos.\ \beta$. Prior igitur in hanc mutatur:

$d =$

$$d = \frac{\sqrt{(c^2n + 2bchq \pm c\sqrt{(c^2n^2 - 4bhn(bnh - cq))})}}{\sqrt{(2n}}$$

Poſterior ad hanc quadraticam deprimitur: $(b^2h^2 - d^2p^2)\,c^2 - 2d^2bhnqc + d^4n^2 = o$, ex qua elicitur: $c = \frac{n(d^2bhq \pm \sqrt{(d^2 - b^2h^2)}}{b^2h^2 - d^2n^2}$. In aequatione pro angulo ACD fit, $m = o$, $n = -1$, et aequatio in hanc mutatur:

$$t = \frac{A^2(d^2 \pm \sqrt{(d^2 - A^4 - d^4 + d^2c^2)})}{A^4 + d^4 - d^2c^2} = tang.\ \beta.$$

Coroll. IX.

§. 543. Quod ſi angulus B ponatur aequalis duobus rectis, lateribus BC, BA in diagonalem AC incidentibus, et trapezio in triangulum ACD abeunte, in aequatione quadratica et quarti gradus corollarii ſecundi fit, $h = o$, et prior in hanc mutatur:

$$d = \frac{c\sqrt{(p^2 + mnq \pm p\sqrt{(1 - (q - 2mn)^2)})}}{n\sqrt{2}},$$

et conſequenter ex hac aequatione fit: latus AD, $c = \frac{dn\sqrt{2}}{\sqrt{(p^2 + 2mn \pm p\sqrt{(1 - (q - 2mn)^2)})}}$. poſterior in hanc abit, ad ſecundum gradum depreſſa et extracta radice:

$$c = \frac{d\sqrt{(p^2 + 2mnq \pm p\sqrt{(1 - (q - 2mn)^2)})}}{m\sqrt{2}},$$

ex hac autem ſequitur eſſe:

$$d = \frac{cm\sqrt{2}}{\sqrt{(p^2 + 2mnq \pm p\sqrt{(1 - (q - 2mn)^2)})}}.$$

Similiter in aequatione quadratica pro angulo A fit

fit $h = o$, et conſequenter aequatio, in hanc mutatur: $fin\ \psi = Q\ \surd(P^2 + Q^2)$, denique in aequatione quadratica pro angulo ACD, fit $A^2 = -c^2\ fin.\ \psi$, ſed quoad formam externam aequatio invariata manet.

Scholion II.

§. 544. Per poſitionem octavi corollarii unum ad triangula ſpectans problema methodo tetragonometrica ſolvitur, quod nullo modo alio ſolvitur, et tunc obtinet, quando quaeritur latus CD; in nullo enim trium triangulorum dantur niſi duo, et in totali ſegmentum ED, lateris BD, et angulus partialis ECD; caeteri caſus etiam ſi nova quadam ratione tetragonometrice ſolvantur, tamen etiam vulgaribus methodis ſolvuntur. Per poſitionem ultimi corollarii etiam ſi novae ſolutiones tetragonometricae prodeant, tamen etiam ſinguli caſus methodis vulgaribus expediuntur.

Scholion III.

§. 545. Aequatio in reſolutione hujus problematis data valet in hypotheſi figurae conſtructae, quando angulus ACD eſt acutus, ſi vero ſit obtuſus valet haec aequatio: $bc\ fin.\ \lambda = -d\ fin.\ (\beta+\psi)\ \surd(c^2 - d^2\ fin.\ \beta^2) + d^2\ fin.\ \beta\ cos.\ (\beta+\psi) + c^2\ fin.\ \psi$. Generalis igitur aequatio pro trapezio directo eſt haec: $b\ c\ fin.\ \lambda = \pm\ d\ fin.\ (\psi \mp \beta)\ \surd(c^2 - d^2\ fin\ \beta^2) \mp d^2\ fin.\ \beta\ cos.\ (\psi \mp \beta) + c^2\ fin.\ \psi$. Quod ſi latus AD cadat intra latus AB exiſtente angulo ACD acuto, obtinet haec aequatio: $bc\ fin.\ \lambda = d\ fin.\ (\beta+\psi)\ \surd(c^2 - d^2\ fin.\ \beta^2) + d^2\ fin.\ \beta\ cos.\ (\beta+\psi) + c^2\ fin.\ \psi$; ſed exiſtente eodem obtuſo,

tuſo, obtinet haec: $b\,c\,\text{ſin.}\,\lambda = d\,\text{ſin.}\binom{\psi-\beta}{\beta-\psi}$ $\sqrt{(c^2-d^2\,\text{ſin.}\,\beta^2)} - d^2\,\text{ſin.}\,\beta\,\text{cos.}\binom{\psi-\beta}{\beta-\psi} + c^2\,\text{ſin.}\,\psi.$

In hoc itaque caſu ſive trapezium ſit partialiter ſive totaliter inverſum valet haec aequatio: $b\,c\,\text{ſin.}\,\lambda = d\,\text{ſin.}\binom{\psi\pm\beta}{\beta-\psi}\sqrt{(c^2-d^2\,\text{ſin.}\,\beta^2)} \pm d^2\,\text{ſin.}\,\beta$ $\text{cos.}\binom{\psi\pm\beta}{\beta-\psi} + c^2\,\text{ſin.}\,\psi.$ Si latus AB cadat intra latus AD, exiſtente angulo ACD acuto, obtinet haec aequatio: $b\,c\,\text{ſin.}\,\lambda = d\,\text{ſin.}\,(\beta-\psi)$ $\sqrt{(c^2-d^2\,\text{ſin.}\,\beta^2)} + d^2\,\text{ſin.}\,\beta\,\text{cos.}\,(\beta-\psi) - c^2\,\text{ſin.}\,\psi.$ Sed exiſtente eodem obtuſo haec habetur: $b\,c\,\text{ſin.}\,\lambda = d\,\text{ſin.}\,(\beta+\psi)\sqrt{(c^2-d^2\,\text{ſin.}\,\beta^2)} - d^2\,\text{ſin.}\,\beta$ $\text{cos.}\,(\beta+\psi) - c^2\,\text{ſin.}\,\psi.$ In hoc itaque caſu, ſive trapezium ſit totaliter ſive partialiter inverſum, valet haec aequatio generalis: $b\,c\,\text{ſin.}\,\lambda = d\,\text{ſin.}(\beta\mp\psi)$ $\sqrt{(c^2-d^2\,\text{ſin.}\,\beta^2)} \pm d^2\,\text{ſin.}\,\beta\,\text{cos.}(\beta\mp\psi) - c^2\,\text{ſin.}\,\psi.$ Aequatio igitur generaliſſima erit hujusmodi: $b\,c\,\text{ſin.}\,\lambda = \pm d$ $\text{ſin.}\binom{\psi\mp\beta}{\beta-\psi}\sqrt{(c^2 - \lambda\,\text{ſin.}\,\beta^2)} \mp d^2\,\text{ſin.}\,\beta$ $\text{cos.}\binom{\psi\mp\beta}{\beta-\psi} \pm c^2\,\text{ſin.}\,\psi.$

Problema XXXIX.

Fig. XXXIX. §. 546. In figura quadrilatera propoſita $ABCD$, inter haec ſex: latus $AB=a$, $BC=b$, $CD=d$, $AD=c$, angulum $A=\psi$; $ACD=\beta$; aequationem invenire

Solutio.

Ex angulis B et D in diagonalem demiſſis perpendiculis Bb, Dd, per antecedentia conſtat

eſſe

eſſe diagonalem $AC = d\ cos.\beta + \surd(c^2 - d^2\ fin.\beta^2)$ et $cos.CAB = \frac{cos.\psi\surd(c^2 - d^2\ fin.\beta^2) + d\ fin.\beta\ fin.\psi}{c}$; conſequenter in triangulo ſiniſtro per (Eucl. II. 11.) pervenio ad hanc aequationem: $BC^2 = AB^2 + AC^2 - 2Ab.\ CD$, et ſymbolis ſubſtitutis habetur: $b^2 = \frac{a^2 + (d\ cos.\beta + \surd(c^2 - d^2\ fin.\beta^2))^2 -}{c}$

$\frac{-(2\ a\ d\ cos.\beta + 2\ a\ \surd c^2 - d^2\ fin.\beta^2)\ cos.\psi}{c}$

$\frac{\surd(c^2 - d^2\ fin.\beta^2) + d\ fin.\beta\ fin.\psi}{c}$, quae aequatio, facta actuali multiplicatione, concinnatione et transpoſitione, fit hujusmodi: $2d\ (c\ cos.\beta - a\ cos.(\psi - \beta))\surd(c^2 - d^2\ fin.\beta^2) = (b^2 - a^2 - c^2 + d^2 - 2d^2\ cos.\beta^2 + 2\ a\ c\ cos.\psi)\ c + 2\ a\ d^2 fin.\beta\ fin.(\psi - \beta)$.

Coroll. I.

§. 547. Sit ad hanc aequationem abbreviandam, $a^2 + c^2 - d^2 + 2d^2 cos.\beta^2 - 2\ a\ c\ cos.\psi = A^2$, $d^2 fin.\beta\ fin.(\psi - \beta) = B^2$, $(c\ cos.\beta - a\ cos.(\psi - \beta)\surd(c^2 - d^2\ fin.\beta^2) = C^2$, et facta ſubſtitutione, erit aequatio abbreviata: $(b^2 - A^2)\ c + 2\ a\ B^2 = 2\ d\ C^2$, ex qua ſtatim habetur pro latere BC haec aequatio: $b = \frac{\surd(A^2 C + 2\ d\ C\ 2\ a\ B^2}{\surd c}$. Ad inveniendum latus AB ſit ad abbreviandum: $b^2 - c^2 + d^2 - 2\ d^2\ cos.\beta^2 = N$, $fin.\beta = m$, $cos.\beta = n$, $fin.(\psi - \beta) = p$, $cos.(\psi - \beta) = q$, $cos.\psi = h$, et $\surd(c^2 - d^2 m^2) = P$, et factis ſubſtitutionibus erit aequatio abbreviata: $(N^2 - a^2 + 2\ a\ c\ h)\ c + 2\ a\ d^2\ m\ p = 2\ d(c\ n - a\ q)P$,

quare transponendo et dividendo habetur:

$$a^2 - \frac{2a(c^2h + d^2mp + Pdq)}{c} = N - 2Pdn,$$

ex qua operatione absoluta obtinetur:

$$a = \frac{c^2h + d^2mp + Pdq \pm \sqrt{((c^2h + d^2mp + Pdq)^2}}{c}$$
$$\frac{+ (N^2 - 2Pdn)^2 c^2)}{c}.$$

Coroll. II.

§. 548. Sit ad abbreviandum $b^2 - a^2 - c^2 + 2ac\, cos. \psi = N^2$, $1 - 2\, cos. \beta^2 = M^2$, $sin. \beta = m$, $cos. \beta = n$, $sin. (\psi - \beta) = p$, $cos. (\psi - \beta) = q$, et factis substitutionibus, fiet aequatio abbreviata: $N^2c + d^2(M^2c + 2amp) = 2d(cn - aq)\sqrt{(c^2 - d^2m^2)}$; sit rursus ad abbreviandum, $M^2c + 2anp = Q$, et $cn - aq = P$, erit aequatio magis abbreviata: $N^2c + d^2Q = 2dP\sqrt{(c^2 - d^2m^2)}$, ex qua pro obtinendo latere CD, habetur, absoluta operatione:

$$d = \frac{\sqrt{(c(2P^2c - N^2Q^2 \pm 2P}}{\sqrt{(Q^2 + 4P^2m^2)}}$$
$$\frac{\sqrt{(P^2c^2 - N^2(Qc + N^2m^2))))}}{\sqrt{(Q^2 + 4P^2m^2)}}.$$

Coroll. III.

§. 549. Ad inveniendum angulum ACD, fit ad abbreviandum $b^2 - a^2 - c^2 + d^2 + 2ac\, cos. \psi = N^2$, ut facta substitutione sit aequatio abbreviata: $N^2c - 2d^2c\, cos. \beta^2 + 2ad^2\, sin. \beta\, sin. (\psi - \beta) =$ $= 2dc\, cos \beta - a\, cos. (\psi - \beta)\sqrt{(c^2 - d^2\, sin. \beta^2)}$, ex qua sumtis quadratis, facta concinnatione, separatione sinuum, et multiplicatione termini N^4

$N^4 c^2$, per *fin.* $\beta^2 +$ *cos.* β^2, ad totam aequationem respectu anguli β homogeneam reddendam, denique divisione totius aequationis per *cos.* β^2 et transpositione, habetur haec aequatio: $(N^4 c^2 - 4 N^2 a c d^2 \cos.\psi + 4 a^2 d^4 - 4 a^2 c^2 d^2 \sin.\psi^2)$ $\tang.\beta^2 + (8 a c^3 d^2 + 4 N^2 a c d^2 - 8 a c d^4 - 8 a^2 c^2 d^2$ $\cos.\psi) \sin.\psi \tang.\beta = 4 c^4 d^4 - 8 a c^3 d^2 \cos.\psi + 4 a^2 c^2 d^2$ $\cos.\psi^2 + 4 N^2 c^2 d^2 - N^4 c^2 - 4 c^2 d^4$. Sit ad abbreviandum coefficiens termini primi $= A$, secundi $= 2 B$ et terminus cognitus $= C$, et factis substitutionibus erit aequatio abbreviata: $A \tang.\beta^2 + 2 B \tang.\beta = C$, ex qua elicitur: $\tang.\beta = \frac{-B \pm \surd(A C + B^2)}{A}$.

Coroll. IV.

§. 550. Ad inveniendum angulum A, sit ad abbreviandum $b^2 - a^2 + d^2 - 2 d^2 \cos.\beta^2 = N^2$, $\surd(c^2 - d^2 \sin.\beta^2) = M$, ut fiat aequatio substitutionibus factis: $N^2 c^2 + 2 a c^2 \cos.\psi + 2 a d^2 \sin.\beta$ $\sin.(\psi - \beta) = 2 M d\,(c \cos.\beta - a \cos.(\psi - \beta)$, posito ad abbreviandum $\sin.\beta = m$, $\cos.\beta = n$, $\sin.\psi = x$, $\cos.\psi = \surd(1 - x^2)$, facta separatione et transpositione habetur: $(2 a c^2 + 2 a d n M - 2 a d^2 m^2)$ $\surd(1 - x^2) = 2 d c n M - N^2 c - 2 a d m (M + n d) x$. Sit jam denuo ad abbreviandum coefficiens membri sinistri aequationis $= A$, terminus absolutus in dextro membro $= B$, et coefficiens termini adfecti $= C$, et habetur aequatio abbreviata hujusmodi: $A \surd(1 - x^2) = B - C X$, ex qua sumtis quadratis et operatione absoluta obtinetur: $x = \frac{B C \pm A \surd(A^2 + C^2 - B^2)}{A^2 + C^2} = \sin.\psi$.

 Schol.

Scholion I.

§. 551. Inter ſex caſus, qui in aequatione continentur, tres ſunt Tetragonometriae proprii, qui tum obtinent, cum ex iis, quae in triangulo dextro invenienda proponuntur, conſtructio figurae quaeritur, quorum caſuum duorum ſolutiones dedi, nimirum pro latere DC et angulo ACD, per aequationes quadraticas etſi ſatis operoſas. Sed pro latere AD, aequationem cum ad ſextum gradum aſcendat, et magna conſtet multitudine terminorum, conſulto praetermiſi, cum in praxi exiguae futura ſit utilitatis. Conſtituunt autem hi caſus in Geometria practica, problemata prorſus nova, utilia et ſatis elegantia. Caeteri tres caſus utrique methodo et tetragonometricae et trigonometricae ſubſunt, ſed ita tamen ut trigonometrice facilius et brevius expediantur, quos tamen evolvi, quia ex aequatione non prolixo calculo ſequuntur.

Coroll. V.

§. 552. Si angulus A ſit aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus et trapezio in triangulum BCD abeunte; aequatio principalis in ſolutione problematis inventa in hanc mutatur:

$$(b^2 - a^2 - c^2 + d^2 - 2 d^2 \cos. \beta^2 - 2 a c) c + 2 a d^2 \sin. \beta^2 = 2 d \cos. \beta \, (a + c) + (c^2 - d^2 \sin. \beta^2),$$

unde ſtatim ſequitur fore pro latere BC,

$$b = \frac{\sqrt{}(((a + c)^2 - d^2 + 2 d^2 \cos. \beta) c - 2 a d^2}{\sqrt{} c}$$

$$\frac{\sin. \beta^2 + 2 d \cos. \beta \, (a + c) \sqrt{} (c^2 - d^2 \sin. \beta^2)}{\sqrt{} c}$$

In

In aequatione quadratica pro latere AB fit, $h=-1$, $p=m$, $q=-n$, quae igitur in hanc paululum mutatur: $a = \frac{d^2 m^2 - c^2 - P d n \pm \sqrt{((d^2 m^2 - c^2 - P d n)^2 + (N^2 - 2 P d n)^2 c)}}{c}$.

In aequatione quadratica pro latere DC fit, $N^2 = b^2 - (a+c)^2$, $p=m$, $q=-n$, $Q = M^2 c + 2 a m^2$, et $P = n(a+c)$, sed aequatio quoad formam externam invariata manet. In aequatione pro angulo ACD fit: $N = b^2 - (a+c)^2 + d^2$, $A = N^4 c^2 + 4 N^2 a c d^2 + 4 a^2 d^4$, $B = o$, $C = 4 c^4 d^4 + 8 a c^3 d^2 + 4 a^2 c^2 d^2 + 4 N^2 c^2 d^2 - N^4 c^2 4 c^2 d^4$, aequatio igitur in hanc mutatur: $tang.\,\beta = \sqrt{\frac{C}{A}}$.

Scholion II.

§. 553. Haec positio unum novum ad triangula spectans solvit problema, quod tunc obtinet quando quaeritur latus DC, cum in hoc casu in triangulis partialibus non dentur nisi duo, neque in totali nisi duo cum angulo partiali ECB; alter casus obtineret siquidem pro inveniendo latere AD solutio tetragonometrica satis commoda ac utilis daretur; caeteri casus etiam si tetragonometrice novis methodis solvantur, tamen methodis vulgaribus facilius expediuntur.

Coroll. VI.

§. 554. Si angulus A sit rectus vel etiam tribus rectis aequalis, cadente vertice A supra

 diago-

diagonalem et trapezio inverſo prodeunte, fit in aequatione pro latere BC, $A^2 = a^2 + c^2 - d^2 + 2d^2 \cos.\beta$, et $\pm d^2 \sin.\beta \cos.\beta = B^2$, et $(c \cos.\beta \pm a \sin.\beta \surd \surd (c^2 - d^2 \sin.\beta) = C^2$, ipſa vero aequatio quoad formam externam invariata manet. In aequatione quadratica pro latere AB fit $p = \pm n$, $q = \mp m$, $h = o$, quare aequatio in hanc mutatur: $a = \frac{\pm (c^2 n - P d) m \pm}{c}$

$$\frac{\surd ((N^2 - 2 P d n)^2 c^2 + m^2 (d^2 n - P d)^2)}{c}$$

In aequatione quadratica pro latere CD fit, $N^2 = b^2 - a^2 - c^2$, et ut prius $p = \pm n$, $q = \mp m$, $Q = M^2 c \pm 2amn$, et $P = cn \pm am$, verum aequatio quoad formam externam invariata manet; in aequatione quadratica pro angulo ACD fit: $b^2 - a^2 - c^2 + d^2 = N^2$, $A = N^4 C^2 + 4a^2 d^4 \mp 4a^2 c^2 d^2$, $2B = 8ac^3 d^2 + 4N^2 acd^2 - 8acd^4$, $C = 4c^4 d^4 + 4N^2 c^2 d^2 - N^4 c^2 - 4c^2 d^4$, ſed quoad formam externam aequatio invariata manet.

Coroll. VII.

§. 555. Si angulus ACD ſit rectus vel etiam aequalis tribus rectis, cadente latere CD, ſupra CB et trapezio inverſo prodeunte; fit in aequatione quadratica pro latere BC, $N = a^2 + c^2 - d^2 - 2ac \cos.\psi$, $B = -d^2 \cos.\psi$, $C = a \sin.\psi \surd (c^2 - d^2)$, ſed aequatio quod ad formam externam invariata manet; in aequatione quadratica pro latere AB fit: $N = b^2 - c^2 + d^2$, $m = \pm 1$, $n = o$, $p = \mp \cos.\psi = \mp h$, $q = \pm \sin.\psi$, et $P = \surd (c^2 - d^2)$, quare aequatio in hanc mutatur:

$a =$

$$a = \frac{c^2h - d^2h \pm Pdq \pm \surd((c^2h - d^2h \pm Pdq)^2 + N^2c^2)}{c};$$

in aequatione quadratica pro latere CD fit, $M^2 = 1$, $m = \pm 1$, $n = 0$, $p = \mp \cos.\psi$, $q = \pm \sin.\psi$, $Q = c - 2ap$, $P = \mp a \sin.\psi = \mp aq$, ipſa vero aequatio quoad formam externam invariata manet. Denique in aequatione pro angulo A fit, $N = b^2 - a^2 + d^2$, $M = \surd(c^2 - d^2)$, et ut prius $m = \pm 1$, $n = 0$, $A = 2ac^2 - 2ad^2$, $B = -NC^2$, $C = \pm 2adM$, ſed quoad formam externam aequatio invariata manet.

Coroll. VIII.

§. 556. Si ponatur angulus ACD evaneſcens abeunte trapezio in triangulum ſiniſtrum ABC, lateribus AD, CD finitae magnitudinis exiſtentibus et in diagonalem AC incidentibus, ita quidem ut ſit $Cb = d\cos.\beta = d$, et conſequenter $AD = Ab$, aequatio in hanc mutatur: $b^2 - a^2 - c^2 - d^2 + 2ac\cos.\psi = 2dc - 2ad\cos.\psi$. Ex hac aequatione quae reſpectu quantitatis c, a ſexto ad ſecundum gradum deſcendit, elicitur latus $AD = c = a\cos.\psi - d \pm \surd(b^2 - a^2\sin.\psi^2)$ et conſequenter latus $CD = d = a\cos.\psi - c \pm \surd(b^2 - a^2\sin.\psi^2)$ et ex eadem ſequitur eſſe: $b = \surd(a^2 + c^2 + d^2 - 2ac\cos.\psi + 2d(c - a\cos.\psi))$ et pro angulo $A\cos.\psi = \frac{a^2 - b^2 + (c+d)^2}{2a(c+d)}$, denique eſſe latus $AB = a = (c+d)\cos.\psi \pm \surd(b^2 - (c+d^2)\sin.\psi^2)$.

Scholion II.

§. 557. Nullum quidem hac positione ad triangula spectans problema novum solvitur, cum licet ad has solutiones nova methodo tetragonometrica sit perventum, tamen ad easdem methodis vulgaribus etiam perveniatur, cum in triangulo ABC, quinque determinationes aequationem ingredientes habeantur, quarum quaecunque incognita ponitur, semper in triangulo tria data habentur, et consequenter etiam singulae quaestiones trigonometrice solvi poterant.

Scholion III.

§. 558. In hypothesi figurae constructae, nimirum quando anguli aequationem ingredientes sunt acuti, valet aequatio proposita. Si manente angulo CAB acuto, sit angulus ACD obtusus, valet haec aequatio: $(b^2-a^2-c^2+d^2-2d^2 \cos.\beta^2 + 2ac \cos.\psi) - 2ad^2 \sin.\beta \sin.(\psi+\beta) = 2d(c \cos.\beta + a \cos.(\psi+\beta) \surd(c^2-d^2 \sin.\beta^2)$; si angulus CAB sit obtusus, sed ACD acutus, valet quoad omnia aequatio in resolutione primo inventa, nisi quod terminus $2ac \cos.\psi$ signo negativo afficiatur. Si denique uterque angulus sit obtusus, valet haec aequatio: $(b^2-a^2-c^2+d^2-2d^2 \cos.\beta^2-2ac \cos.\psi) c + 2ad^2 \sin.\beta \sin.(\psi+\beta) = 2d(c \cos.\beta - a \cos.(\psi+\beta) \surd(c^2-d^2 \sin.\beta)$. Si latus AB cadat intra latus AD, quomodocunque trapezium sit inversum, iidem casus recurrunt, et aequationes in iisdem casibus angulorum eaedem manent. Si latus AD cadat intra latus AB, quomodocunque trapezium sit inversum, dummodo angulus uterque existat acutus, valet

valet ſine omni mutatione aequatio primo inventa. Si manente angulo CAB acuto, ſit angulus ACD obtuſus, in trapezio inverſo valet aequatio caſus ejusdem prius inventa in trapezio directo, in caſu angulorum tertio valet haec aequatio: $(b^2-a^2-c^2+d^2-2d^2 cos.\beta^2-2ac\,cos.\psi)$ $c+2ad^2 fin.\beta\, fin.(\psi+\beta)=-2d\,(c\,cos.\beta-a\,cos.(\psi+\beta))\,\surd(c^2-d^2 fin.\beta^2)$. Denique in caſu quarto valet haec aequatio: $(b^2-a^2-c^2+d^2-2d^2 cos.\beta^2-2ac\,cos.\psi)\,c-2ad^2 fin.\beta\, fin.(\psi-\beta)$ $=2d\,(c\,cos.\beta-a\,cos.(\psi-\beta))\,\surd(c^2-d^2 fin.\beta^2)$. Generaliſſima igitur aequatio eſt hujusmodi: $(b^2-a^2-c^2+d^2-2d^2 cos.\beta^2 \pm 2ac\,cos.\psi)$ $c \mp 2ad^2 fin.\beta\, fin.(\psi\pm\beta) = \mp 2d$ $(c\,cos.\beta \pm a\,cos.(\psi+\beta))\,\surd(c^2-d^2 fin.\beta^2)$.

Problema XL.

§. 559. In figura quadrilatera propoſita $ABCD$, inter haec ſex: latus $AB=a$, $CD=d$, $AD=c$, angulum $A=\psi$, $B=\lambda$, et $ACD=\beta$; aequationem invenire.

Solutio.

Cum ſit per andecedentia diagonalis AC $=\surd(c^2-d^2 fin.\beta^2)+d\,cos.\beta$ et Fig. XL.

$$fin.CAB=\frac{fin.\psi\surd(c^2-d^2 fin.\beta^2)-d\,fin.\beta\, cos.\psi}{c},\ \text{et}$$

$$cos.CAB=\frac{cos.\psi\surd(c^2-d^2 fin.\beta^2)+d\,fin.\beta\, fin.\psi}{c},$$

ex quibus et ex ſinu coſinuque anguli B obtinetur: $fin.\ ACB=$

$$=\frac{fin.(\lambda+\psi)\surd(c^2-d^2 fin.\beta^2)-d\,fin.\beta\, cos.(\lambda+\psi)}{c}$$

 in

in triangula finiſtro pervenio ſtatim ad hanc analogiam: $AB : \mathit{fin.}\, ACB = AC : \mathit{fin.}\, B$, h. e:

$$a : \frac{\mathit{fin.}(\lambda+\psi)\surd(c^2-d^2\mathit{fin.}\beta^2)-d\,\mathit{fin.}\beta\cos.(\lambda+\psi)}{c} =$$

$=\surd(c^2-d^2\mathit{fin.}\beta^2)+d\cos.\beta : \mathit{fin.}\lambda$, unde ſequitur fore: $ac\,\mathit{fin.}\lambda = d\,\mathit{fin.}(\lambda+\psi-\beta)\surd(c^2-d^2\mathit{fin.}\beta^2) - d^2\mathit{fin.}\beta\cos(\lambda+\psi-\beta)+c^2\mathit{fin.}(\lambda+\psi)$, Q. E. I.

Coroll. I.

§. 560. Ex hac aequatione ſtatim incurrit in oculos eſſe latus $AB = a = \frac{d\,\mathit{fin.}\,(\lambda+\psi-\beta)}{c\,\mathit{fin.}\lambda}$

$$\frac{\surd(c^2-d^2\mathit{fin.}\beta^2)-d^2\mathit{fin.}\beta\cos.(\lambda+\psi-\beta)+c^2\mathit{fin.}(\lambda+\psi)}{c\,\mathit{fin.}\lambda}.$$

Ut ad aequationem quarti gradus, pro obtinendo latere AD, perveniatur, ſit ad abbreviandum, $\mathit{fin.}\beta = m$, $\mathit{fin.}\lambda = h$, $\mathit{fin.}(\lambda+\psi) = n$, $\mathit{fin.}(\lambda+\psi-\beta) = p$, et $\cos.(\lambda+\psi-\beta) = q$, et factis ſubſtitutionibus et tranſpoſitione, habetur aequatio abbreviata: $ach + d^2mq - c^2n = dp\surd(c^2-d^2m^2)$, ex qua evoluta ſumtis quadratis, facta concinnatione, tranſpoſitione et ordinatione, obtinetur haec aequatio quarti gradus completa:

$$c^4 - \frac{2ac^3h}{n} + \left(\frac{a^2h^2}{n^2} - \frac{d^2p^2}{n^2} - \frac{2d^2mq}{n}\right)c^2 + \frac{2acd^2hmq}{n^2} + \frac{d^4m^2}{n^2} = o,$$

iisdem nominibus ad abbreviandum adhibitis invenitur eſſe latus CD

$$d = \frac{\surd(c^2p^2 - 2cmq(ah-cn)) \pm cp}{m\surd 2}$$

$$\frac{\surd(c^2p^2-4(ah-cn)(cmq+m^2(ah-cn)))}{m\surd 2}.$$

Corol

Coroll. II.

§. 561. Pro angulo B obtinendo facta separatione sinuum et operatione absoluta, habetur: $tang.\ \lambda =$

$$= \frac{c^2 fin.\psi - d^2 fin.\beta\, cos.(\psi-\beta) + d\, fin.(\psi-\beta)\sqrt{(c^2-d^2 fin.\beta^2)}}{c(a-c\,cos.\psi) - d\,(cos.(\psi-\beta)\sqrt{(c^2-d^2 fin.\beta^2)} + d^2 fin\,\beta\, fin.(\psi-\beta))},$$

vel etiam $tang.\ \lambda =$

$$= \frac{c^2 fin\,\psi\, fec.(\psi-\beta) - d^2 fin.\beta + d\, tang.(\psi-\beta)\sqrt{(c^2-d^2 fin.\beta^2)}}{c(a-cos.\psi)\, fec.(\psi-\beta) - d^2(\sqrt{(c^2-d^2 fin.\beta^2)} + d\, fin.\beta\, tang.(\psi-\beta))},$$

Si ad abbreviandum ponatur $a\,c\,fin.\ \lambda - c^2$ $fin.\ (\lambda+\psi) = A^2$, ut sit facta substitutione et transpositione, $A^2 + d^2\, fin.\ \beta.\, cos.(\lambda+\psi-\beta) = d$ $fin.(\lambda+\psi-\beta)\sqrt{(c^2 - d^2 fin.\beta^2)}$ erit sumtis quadratis, facta transpositione et concinnatione, $A^4 + 2A^2d^2 fin.\,\beta\, cos.(\lambda+\psi-\beta) + d^4 fin.\,\beta^2 = d^2c^2$ $fin.\ (\lambda+\psi-\beta)^2$, multiplicato termino A^4 per $fin.\ \beta^2 + cos.\ \beta^2$, ad aequationem respectu anguli β homogeneam reddendam, sinibus separatis, et in sinistram partem aequationis translatis terminis omnibus, qui quadrato $fin.\ \beta^2$ vel etiam $fin.\ \beta\ cos.\ \beta$ afficiuntur, caeteris in dextram, facta divisione per $cos.\ \beta^2$, et posito ad abbreviandum: $fin.\ (\lambda+\psi) = m,\ cos.\ (\lambda+\psi) = n,$ $tang.\,\beta = t$, prodibit haec aeqatio qradratica omni operatione absoluta:

$$t = \frac{-d^2n(A^2+c^2m) \pm \sqrt{(d^4n^2(A^2+c^2m)^2}}{A^4+d^4+2A^2d^2m-d^2c^2n^2}$$

$$\frac{+(d^2c^2m^2-A^4)(A^4+d^4+2A^2d^2m-d^2c^2n^2))}{A^4+d^4+2A^2d^2m-d^2c^2n^2}$$

Coroll.

Coroll. III.

§. 562. Pro inveniendo angulo A, posito ad abbreviandum $fin.\, \beta = h$, $fin.\, (\lambda - \beta) = m$, $cos.\, (\lambda - \beta) = n$, $fin.\, \lambda = p$, $cos.\, \lambda = q$, $\sqrt{(c^2 - d^2 h)} = M$, $fin.\, \psi = B$, $cos.\, \psi = \sqrt{(1 - x^2)}$, facta sinuum separatione, substitutione et transpositione habetur haec aequatio abbreviata: $acp - (Mdn + d^2 hm + c^2 q)\, x = (Mdm + c^2 p - d^2 hn) \sqrt{(1 - x^2)}$. Sit denuo ad abbreviandum coefficiens ipsius $x = A^2$, et coefficiens ipsius $\sqrt{(1 - x^2)} = B^2$, erit aequatio abbreviata: $acp - A^2 x = B^2 \sqrt{(1 - x^2)}$, ex qua operatione absoluta elicitur:

$$x = \frac{A^2 acp \pm B^2 \sqrt{(A^4 + B^4 - a^2 c^2 p^2)}}{A^4 + B^4} = fin.\, \psi.$$

Scholion 1.

§. 563. Inter sex casus qui in aequatione continentur, duo sunt Tetragonometriae proprii, qui tunc obtinent, quando ex alterutro latere in triangulo dextro inveniendo, figura construenda proponitur, qui casus in Geometria practica constituunt duo problemata nova utilia juxta ac satis pulchra. Caeteri casus quatuor utrique methodo et tetragonometricae et trigonometricae subsunt, sed ita ut trigonometrice fere facilius solvantur, quos tamen evolvi, quia ex aequatione non difficulter deducuntur.

Coroll. IV.

§. 564. Si anguli A et D ponantur simul aequales duobus rectis, ut prodeat trapezium paral-

parallelarum basium AD, BC, aequatio pro latere AB in hanc mutatur:

$$a = \frac{d\, \mathit{sin.}\, \beta \sqrt{(c^2 - d^2 \mathit{sin.}\, \beta^2)} + d^2\, \mathit{sin.}\, \beta\, \mathit{cos.}\, \beta}{c\, \mathit{sin.}\, \lambda},$$

vel etiam

$$a = \frac{d\, \mathit{sin.}\, \beta \left(\sqrt{(c^2 - d^2 \mathit{sin.}\, \beta^2)} + d^2\, \mathit{sin.}\, \beta\, \mathit{cos.}\, \beta\right)}{c\, \mathit{sin.}\, \psi},$$

in aequatione quarti gradus pro latere AD fit, $n = o$, $p = m$, $q = -\mathit{cos.}\, \beta$, evanescentibus omnibus terminis in quibus occurrit n, aequatio ad hanc quadraticam deprimitur: $(a^2 h^2 - d^2 p^2)$ $c^2 - 2 a c d^2 h m q + d^4 m = o$, ex qua, operatione absoluta, elicitur:

$$c = \frac{d^2 m \left(-a h q \pm m \sqrt{(d^2 - a^2 h^2)}\right)}{a^2 h^2 - d^2 m^2},$$ sed aequatio quadratica pro latere CD in hanc mutatur:

$$d = \frac{\sqrt{(c^2 m - 2 a c h q \pm c \sqrt{(c^2 m^2 - 4 a h m (a h + c q))})}}{\sqrt{2 m}}.$$

In aequatione quadratica pro angulo ACD fit $A^2 = a c\, \mathit{sin.}\, \lambda$, $m = o$, $n = -1$, et aequatio restitutis valoribus in hanc mutatur:

$$t = \frac{a c\, \mathit{sin.}\, \lambda \left(d^2 \pm c \sqrt{(d^2 - a^2 \mathit{sin.}\, \lambda.)}\right)}{a^2 c^2 \mathit{sin.}\, \lambda^2 + d^4 - d^2 c^2}.$$

Coroll. V.

§. 565. Si angulus A sit rectus vel etiam aequalis tribus rectis, cadente vertice A supra diagonalem BD, et trapezio inverso prodeunte, aequatio pro latere AB in hanc mutatur:

$a =$

$a = \frac{d\ cos.(\lambda-\beta) \sqrt{(c^2 - d^2 fin.\beta^2)} \pm d^2 fin.\beta}{c\ fin.\lambda}$

$\frac{fin.(\lambda-\beta) \pm c^2 cos.\lambda}{c\ fin.\lambda}$. In aequatione quarti gradus pro latere AD et quadratica pro latere CD fit, $n = \pm cos.\lambda$, $p = \pm cos.(\lambda-\beta)$, $q = \mp fin.(\lambda-\beta)$, prior igitur paululum in hanc mutatur: $c^4 \mp \frac{2ac^3h}{n} + \left(\frac{a^2h^2}{n^2} - \frac{d^2p}{n^2} + \frac{2d^2mq}{n}\right)$

$c^2 \mp \frac{2acd^2hmq}{n^2} + \frac{d^4m^2}{} = o$, posterior in hanc abit:

$d = \frac{\sqrt{(c^2p^2 - 2cmq(cn \mp an)} \pm cp}{m\sqrt{2}}$

$\frac{\sqrt{(c^2p^2 - 4(ah \mp cn)(m^2(ah \mp cn) \mp cmq)))}}{m\sqrt{2}}$.

Aequatio pro angulo B in hanc mutatur:

$tang.\lambda = \frac{d\ cos.\beta \sqrt{(c^2 - d^2 fin.\beta^2)} \pm d^2 fin.\beta^2 \pm c^2}{ac \pm d\ fin.\beta \sqrt{(c^2 - d^2 fin.\beta^2)} \mp d^2 fin.\beta\ cos.\beta}$.

In aequatione pro angulo ACD fit, $A^2 = ac\ fin.\lambda \mp c^2 cos.\lambda$, $m = \pm cos.\lambda$, $n = \mp fin.\lambda$, quae igitur in hanc paululum mutatur:

$t = \frac{d^2n(c^2m \pm A^2) \pm \sqrt{(d^4n^2(A^2 \pm c^2m)^2}}{A^4 + d^4 \pm 2A^2d^2m - d^2c^2n^2}$

$\frac{+(d^2c^2m^2 - A^4)(A^4 + d^4 \pm 2A^2d^2m - d^2c^2n^2))}{A^4 + d^4 \pm 2A^2d^2m - d^2c^2n^2}$.

Coroll. VI.

§. 566. Si ponatur angulus B rectus vel etiam aequalis tribus rectis, cadente vertice B ad dextram diagonalis AC et trapezio inverso pro-

prodeunte, aequatio pro latere AB in hanc mutatur: $a = \frac{d \cos. (\psi - \beta) \sqrt{(c^2 - d^2 \sin. \beta^2)}}{c}$ $\frac{+ d \sin. \beta \sin. (\psi - \beta) + c^2 \cos. \psi}{c}$. In aequatione quarti gradus pro latere AD et quadratica pro latere DC fit, $h = \pm 1$, $n = \pm \cos. \psi$, $p = \pm \cos. (\psi - \beta)$, $q = \mp \sin. (\psi - \beta)$, quarum prior in hanc paululum mutatur:

$$c^4 - \frac{2ac^3}{n} + \left(\frac{a^2}{n^2} - \frac{d^2 p^2}{n^2} + \frac{2d^2 mq}{n}\right)c^2 - \frac{2acd^2 hmq}{n^2}$$

$$+ \frac{d^4 m^2}{n^2} = 0,$$ sed posterior in hanc abit:

$$d = \frac{\sqrt{(c^2 p^2 + 2cmq(a - cn)} \pm cp}{m\sqrt{2}}$$
$$\frac{\sqrt{(c^2 p^2 - 4(a - cn(m^2(a - cn) \mp cmq)))}}{m\sqrt{2}}.$$

In aequatione pro angulo ACB fit $A^2 = \pm ac \mp c^2 \cos. \psi$, $m = \pm \cos. d$, $n = \mp \sin. \lambda$, quae igitur in hanc paululum mutatur:

$$t = \frac{d^2 n (A^2 + c^2 m) \pm \sqrt{(d^4 n^2 (A^2 \pm c^2 m)^2}}{A^4 + d^4 \pm 2A^2 d^2 m - d^2 c^2 n^2}$$
$$\frac{+ (d^2 c^2 m^2 - A^4)(A^4 + d^4 \pm 2A^2 d^2 m - d^2 c^2 n^2))}{A^4 + d^4 \pm 2A^2 d^2 m - d^2 c^2 n^2}$$

In aequatione pro angulo A fit, $m = \pm \cos. \beta$, $n = \mp \sin. \beta$, $p = \pm 1$, $q = 0$, $A^2 = \mp (Mdn - d^2 hm)$, $B^2 = \mp (d^2 hn - Mdm - c^2)$, quae igitur in hanc paululum mutatur:

$$x = \frac{A^2 ac \pm B^2 \sqrt{(A^4 \pm B^4 - a^2 c^2)}}{A^4 + B^4}.$$

Coroll.

Coroll. VII.

§. 567. Si angulus ACD ponatur rectus vel etiam aequalis tribus rectis, cadente latere DC, ſupra latus BC, prodeunte trapezio totaliter vel partialiter inverſo, aequatio pro latere AB in hanc mutatur: $$a = \frac{d \cos.(\lambda+\psi) \surd(c^2 - d^2) - d^2 \sin.(\lambda+\psi) + c^2 \sin.(\lambda+\psi)}{c \sin.\lambda}.$$ In aequatione quarti gradus pro latere AD, et quadratica pro latere DC fit, $m = \pm 1$, $p = \mp \cos.(\lambda+\psi)$, $q = \pm \sin.(\lambda+\psi)$, quarum prior in hanc paululum mutatur: $$c^4 - \frac{2ac^3h}{n} + \left(\frac{a^2h^2}{n^2} - \frac{d^2p^2}{n^2} - \frac{2d^2q}{n}\right)c^2 + \frac{2acd^2hq}{n^2} + \frac{d^4m^4}{n^2} = 0,$$ poſterior in hanc: $$d = \frac{\surd(c^2p^2 - 2cq(ah - cn) \pm cp}{\surd 2}\,\frac{\surd(c^2p^2 - 4(ah - cn)ah - cn + cq)))}{\surd 2};$$ aequatio pro angulo B in hanc mutatur: $$\tan g.\lambda = \frac{c^2 \sin.\psi \mp d^2 \sin.\psi \mp d \cos.\psi \surd(c^2 - d^2)}{c(a - \cos.\psi) \mp d \sin.\psi \surd(c^2 - d^2 + d \cos.\psi};$$ in aequatione pro angulo A fit, $h = \pm 1$, $n = \mp \sin.\lambda = \mp p$, $m = \pm \cos.\lambda = \pm q$, $M = \surd(c^2 - d^2)$, $A = d^2m \mp Mdn \pm c^2m$, $B^2 = d^2n \pm Mdm \mp c^2n$, et aequatio in hanc paululum mutatur: $$x = \frac{\pm A^2acn \pm B^2 \surd(A^4 + B^4 - a^2c^2n^2)}{A^4 + B^4}.$$

Coroll.

Coroll. VIII.

§. 568. Si angulus A sit aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus et trapezio in triangulum BCD abeunte, fit segmentum EB lateris $BD =$

$$a = \frac{d\ sin.\ (\lambda-\beta)\ \sqrt{(c^2 - d^2\ sin.\ \beta^2)} + d^2\ sin.\ \beta^2}{c\ sin.\ \lambda}$$

$$\frac{cos.\ (\lambda-\beta) - c^2\ sin.\ \lambda}{c\ sin.\ \lambda};$$ in aequatione quarti gradus pro latere DA et quadratica pro latere DC fit, $n = -h$, $p = -sin.\ (\lambda-\beta)$, $q = -cos.\ (\lambda-\beta)$, quarum prior in hanc paululum mutatur:

$$c^4 + 2ac^3 + \left(a^2 - \frac{d^2p^2}{n^2} - \frac{d^2mq}{n}\right)c^2 + \frac{2acd^2m^2q}{n} + \frac{d^4m^2}{n^2} = 0,$$

posterior in hanc abit:

$$d = \frac{\sqrt{(c^2p^2 \pm 2cmnq(a \pm c)} \pm cp}{m\sqrt{2}}$$

$$\frac{\sqrt{(c^2p^2 - 4cn(a+c)(m^2n(a+c) - cmq)))}}{m\sqrt{2}}.$$

Pro angulo B aequatio in hanc mutatur:

$$tang.\ \lambda = \frac{d^2\ sin.\ \beta\ cos.\ \beta + d\ sin.\ \beta\ \sqrt{(c^2 - d^2\ sin.\ \beta^2)}}{c(a+c) + d\ cos.\ \beta\ \sqrt{(c^2 - d^2\ sin.\ \beta^2)} - d^2\ sin.\ \beta^2}$$

vel etiam $tang.\ \lambda =$

$$= \frac{d^2\ cos.\ \beta + d\ \sqrt{(c^2 - d^2\ sin.\ \beta^2)}}{(a+c)\ c\ cosec.\ \beta + d\ cot.\ \beta\ \sqrt{(c^2 - d^2\ sin.\ \beta^2)} - d^2\ sin.\ \beta},$$

in aequatione pro angulo ACD fit: $A^2 = (a+c)\ c\ sin.\ \lambda$, $m = -sin.\ \lambda$, $n = -cos.\ \lambda$,

 et

et aequatio in hanc paululum mutatur:

$$t = \frac{d^2 n (A^2 - c^2 m) \pm \sqrt{} (d^4 n^2 (A^2 - c^2 m)^2}{A^4 + d^4 - 2 A^2 d^2 m - d^2 c^2 n^2}$$

$$\frac{+ (d^2 c^2 m^2 - A^4)(A^4 + d^4 - 2 A^2 d^2 m - d^2 c^2 n^2))}{A^4 + d^4 - 2 A^2 d^2 m - d^2 c^2 n^2}.$$

Scholion II.

§. 569. Positione in hoc ultimo corollario adhibita, resolvuntur duo nova problemata ad triangula spectantia, quando quaeritur vel latus CD, vel DE segmentum lateris AB, cum neque in partialibus, neque in totali triangulo, dentur nisi duo, una cum segmento BE in casu posteriore, et angulo partiali ACD in casu priore, caeteri casus licet tetragonometrice nova methodo solvantur, tamen etiam trigonometrice expediuntur.

Coroll. IX.

§. 570. Si ponatur angulus $B = ACD$ aequatio pro latere AB in hanc mutatur:

$$a = \frac{d.\,fin.\,\psi \sqrt{} (c^2 - d^2\,fin.\,\beta^2) - d^2\,fin.\,\beta}{c\,fin.\,\beta}$$

$$\frac{cos.\,\psi + c^2\,fin.\,(\lambda + \psi)}{c\,fin.\,\beta},$$ angulo λ eliminato,

vel etiam β eliminato: $$a = \frac{d\,fin.\,\psi \sqrt{} (c^2 - d^2\,fin.\,\lambda^2)}{c\,fin.\,\lambda}$$

$$\frac{- d^2\,fin.\,\lambda\,\,cos.\,\psi + c^2\,fin.\,(\lambda + \psi)}{c\,fin.\,\lambda}.$$

In

In aequatione quarti gradus pro latere AD et quadratica, pro latere DC fit, $h = m$, $p = fin.\ \psi$, $q = cos.\ \psi$ quarum illa in hanc abit: $c^4 - \frac{2ac^3m}{n} + (\frac{a^2m^2}{n^2} - \frac{d^2p^2}{n^2} - \frac{2d^2mq}{n})$ $c^2 + \frac{2acd^2m^2q}{n^2} + \frac{d^4m^2}{n^2} = o$,

haec autem etiam in hanc abit:

$$d = \frac{\sqrt{(c^2p^2 - 2mq(am - cn)} \pm cp}{m\sqrt{2}}$$
$$\frac{\sqrt{(c^2p^2 - 4(am - cn)(cmq + m^2(am - cn)))}}{m\sqrt{2}}$$

In aequatione pro angulo A fit, $m = o$, $n = +1$, $p = h$, $q = cos.\ \lambda = cos.\ \beta$, $M = \sqrt{(c^2 - d^2h^2)} = \sqrt{(c^2 - d^2p^2)}$, fed aequatio quoad formam externam invariata manet, nifi quod h pro p fubftitui poffit. Quod fi ponatur angulus $ACD = A$, aequatio pro latere AB, in hanc abit angulo ψ eliminato:

$$a = \frac{d\sqrt{(c^2 - d^2 fin.\ \beta^2)} - d^2 fin.\ \beta\ cot.\ \lambda + c}{c\ fin.\ \lambda}$$

$\frac{fin.\ (\lambda + \beta)}{c\ fin.\ \lambda}$. In aequatione quarti gradus pro latere AD et quadratica pro latere CD fit, $m = fin.\ \beta = fin.\ \psi$, $n = fin.\ (\lambda + \psi) = fin.\ (\lambda + \beta)$, $p = h$, $q = cos.\ \lambda$, quarum itaque prior in hanc paululum mutatur: $c^4 - 2ac^3h + (\frac{a^2h^2}{n^2} - \frac{d^2h^2}{n^2} - 2d^2mq)$

$c^2 + \frac{2acd^2hmq}{n^2} + \frac{d^4m^2}{n^2} = o$, pofterior in hanc abit:

 $d =$

$$d = \frac{\sqrt{(c^2 h^2 - 2cmq(ah - cn)} \pm ch}{m\sqrt{2}}$$

$$\frac{\sqrt{(c^2 h^2 - 4(ah - cn)(cmq + m^2(ah - ac)))}}{m\sqrt{2}}$$

Aequatio pro angulo B in hanc mutatur: $tang.\ \lambda = (c^2 - d^2)\ fin.\ \psi : ((a - c\cos.\psi)c - d\sqrt{(c^2 - d^2\ fin.\ \psi^2)})$.

Scholion III.

§. 571. Aequatio fupra inventa locum habet in hypothefi figurae conftructae, fcilicet fi angulus ACD fit acutus; fi vero fit obtufus ut longitudo diagonalis fit $\sqrt{(c^2 - d^2\ fin.\ \beta^2)} - d\cos.\beta$ obtinet haec aequatio: $ac\ fin.\ \lambda = -d\ fin.(\beta + \lambda + \psi)\sqrt{(c^2 - d^2 fin.\ \beta^2)} + d^2 fin.\ \beta\ \cos.(\beta + \lambda + \psi) + c^2 fin.(\lambda + \psi)$. Si latus AD cadat intra latus AB, quomodocunque trapezium fit inverfum, exiftente angulo ACD acuto obtinet haec aequatio: $ac\ fin.\ \lambda = fin.\ (\beta + \lambda + \psi)\sqrt{(c^2 - \lambda^2 fin.\ \beta^2)} + d^2 fin.\ \beta\ \cos.(\beta + \lambda + \psi) + c^2 fin.\ (\lambda + \psi)$. Sed exiftente angulo ACD obtufo obtinet haec: $ac\,fin.\lambda = fin.(\lambda + \psi - \beta)\sqrt{(c^2 - d^2 fin.\beta^2)} - d^2 fin.\beta\cos.(\lambda + \psi - \beta) + c^2 fin.(\lambda + \psi)$. Si latus AD cadat extra latus AB, quomodocunque trapezium fit inverfum, exiftente angulo ACD acuto, obtinet haec aequatio: $ac\ fin.\ \lambda = d\ fin.(\beta + \lambda - \psi)\sqrt{(c^2 - d^2 fin.\beta^2)} + d^2 fin.\ \beta$ cos.

$cos. (\beta + \lambda - \psi) + c^2 fin. (\lambda - \psi)$. Sed exiſtente eodem obtuſo habetur haec: $a c\, fin. \lambda = d\, fin. (\lambda - \beta - \psi) \surd (c^2 - d^2\, fin.\, \beta^2) - d^2\, fin.\, \beta\, cos.\, (\lambda - \beta - \psi) + c^2\, fin. (\lambda - \psi)$, ubi notandum etiam evenire poſſe, ut angulus $\lambda - \beta - \psi$ fiat negativus, in quo caſu rectius ſcriberetur aequatio hoc modo: $a c\, fin.\, \lambda = d\, fin. (\beta + \psi - \lambda) \surd (c^2 - d^2\, fin.\, \beta^2) \mp d^2\, fin.\, \beta\, cos. (\lambda \mp \psi \pm \beta) + c^2\, fin. (\lambda \pm \psi)$, pro hoc itaque problemate aequatio generaliſſima eſt hujusmodi: $a c\, fin.\, \lambda = d\, fin. (\lambda \mp \psi \pm \beta) \surd (c^2 - d^2\, fin.\, \beta^2) \pm d^2\, fin.\, \beta\, cos. (\lambda \mp \psi \pm \beta) + c^2\, fin. (\lambda \pm \psi)$.



CAPUT XII.

Continens duo problemata sextae sive ultimae classis particularis sub posteriore principali contentae.

Problema XLI.

Fig. XLI. §. 572. In figura quadrilatera proposita $ABCD$, inter haec sex: latus $AB=a$, $BC=b$, $AD=c$, angulum $A=\psi$, $B=\lambda$, $ACD=\beta$, aequationem invenire

Solutio.

1) In triangulo sinistro ab angulo C in latus AB demisso perpendiculo Cc, est per (Eucl. II. 11.) diagonalis $AC = \sqrt{(a^2+b^2-2ab\, cos.\lambda)} = h$ brevitatis gratia, hinc in eodem triangulo formo hanc analogiam $AC\,(h) : fin.\, B\,(fin.\,\lambda) = BC\,(b) : fin.\, CAB = \frac{b\, fin.\,\lambda}{h}$, et hinc $cos.\, CAB = \sqrt{(h^2 - b^2 fin.\,\lambda^2)}$.

2) Ex his et ex sinu cosinuque anguli A formo sinum et cosinum anguli CAD et invenio:

$$fin.\, CAD = \frac{fin.\psi \sqrt{(h^2-b^2 fin.\lambda^2)} - b\, fin.\lambda\, cos.\psi}{h}, \text{ et}$$

$$cos.\, CAD = \frac{cos.\psi \sqrt{(h^2-b^2 fin.\lambda^2)} + b\, fin.\lambda\, fin.\psi}{h}.$$

3) Ex

3) Ex his et ex ſinu coſinuque anguli ACD, formo ſinum anguli D, quem invenio eſſe:

$$= \frac{fin.\,\beta\, cos.\,\psi \sqrt{(h^2 - b^2 fin.\,\lambda^2)} + b\, fin\,\beta\, fin.\,\lambda\, fin.\,\psi}{h}$$

$$+ \frac{cos.\,\beta\, fin.\,\psi \sqrt{(h^2 - b^2 fin.\,\lambda^2)} - b\, cos.\,\beta\, fin.\,\lambda\, cos.\,\psi}{h}$$

$$= \frac{fin.\,(\beta + \psi) \sqrt{(h^2 - b^2 fin.\,\lambda^2)} - b\, fin.\,\lambda}{h}$$

$\frac{cos.\,(\beta + \psi)}{h} = fin.\,D.$ His factis in triangulo dextro pervenio ad hanc analogiam, $AC : fin.\,D = AD : fin.\,ACD$, h. e. $h^2 fin.\,(\beta + \psi) \sqrt{(h^2 - d^2 fin.\,\lambda^2)} - b\, fin.\,\lambda$ $cos.\,(\beta + \psi) = c : fin.\,\beta$, conſequenter habetur haec aequatio: $h^2 fin.\,\beta = c\, fin.\,(\beta + \psi) \sqrt{(h^2 - b^2 fin.\,\lambda^2)}$ $- b\, c\, fin.\,\lambda\, cos.\,(\beta + \psi)$, Q. E. I.

Scholion I.

§. 573. Re accuratius perpenſa aequatio inventa poteſt ab omni irrationalitate liberari, eſt enim $\sqrt{(h^2 - b^2 fin.\,\lambda^2)} = \sqrt{(a^2 + b^2 \mp 2\,a\,b}$ $cos.\,d - b^2 fin.\,\lambda^2) = \sqrt{(a^2 \pm 2\,a\,b\, cos.\,\lambda + b^2 cos.\,\lambda^2)}$ $= a \mp b\, cos.\,d$, quare fiet aequatio inventa: $(a^2 + b^2 \mp 2\,a\,b\, cos.\,\lambda)\, fin.\,\beta = c\, fin.\,(\beta + \psi)$ $(a \mp b\, cos.\,\lambda) - b\,c\, fin.\,\lambda\, cos.\,(\beta + \psi)$, et conſequenter habetur haec aequatio abbreviata ab omni irrationalitate libera: $(a^2 + b^2 \mp 2\,a\,b\, cos.\,\lambda)\, fin.\,\beta =$ $\pm\, a\, c\, fin.\,(\beta + \psi) \mp b\, c\, fin.\,(\beta + \psi \pm \lambda)$. Sed animadvertendum obtinente ſigno affirmativo termini prioris dextri membri, obtinere poſſe

 ſignum

signum utrumque termini sequentis, sed obtinente signo negativo termini illius, non nisi hujus negativum obtinere.

Coroll. I.

§. 574. Ex hac aequatione statim liquet esse: $c = (a^2 + b^2 \mp 2\,a\,b \;cos.\; d)\; sin.\; \beta : \pm (a\; sin.\; (\beta + \psi) \mp b\; sin.\; (\beta + \psi \pm \lambda))$. Sit pro inveniendo latere AB ad abbreviandum: $sin.\; \beta = k$, $sin.\; (\beta + \psi) = p$, $cos.\; \lambda = n$, $sin.\; (\beta + \psi \pm \lambda) = q$, et factis substitutionibus fiet aequatio abbreviata: $(a^2 + b^2 \pm 2\,a\,b\,n) = \pm (a\,c\,p \mp b\,c\,q)$, ex qua operatione ad finem perducta habetur:

$$a = \frac{2bnk \pm cp \pm \sqrt{(2bnk \pm cp)^2 - 4bk(bk \pm cq)}}{2k}.$$

Similiter invenietur esse: latus $BC =$

$$b = \frac{2ank \mp cq \pm \sqrt{(2ank \mp cq)^2 - 4ak(ak \pm cp)}}{2k}.$$

Pro inveniendo angulo ACD habetur haec aequatio: $tang.\; \beta =$

$$= \frac{c(a\; sin.\; \psi \pm b\; cos.(\psi \pm \lambda))}{a^2 + b^2 \mp 2ab\; cos.\; \lambda \mp c(a\; cos.\; \psi \pm b\; cos.(\psi \pm \lambda))}.$$

Coroll. II.

§. 575. Ad angulum B inveniendum sint ad abbreviandum nomina in corollario praecedenti usurpata, et praeterea $sin.\; \lambda = x$, $cos.\; \lambda = \sqrt{(1 - x^2)}$, facta substitutione et transpositione, habetur haec aequatio abbreviata: $(a^2 + b^2)k \mp acp \pm bcqx = b(2ak - cp)\sqrt{(1 - x^2)}$.

Sit

Sit etiam ad abbreviandum terminus abſolutus ſiniſtri membri $= A^2$ et coefficiens datus dextri membri $= B^2$, erit facta ſubſtitutione aequatio abbreviata: $A^2 \pm b c q x = B^2 \sqrt{(1-x)}$, ex qua operatione abſoluta obtinetur haec aequatio:

$$x = \frac{\pm A^2 b c q \pm B^2 \sqrt{(b^2 c^2 q^2 + B^4 - A^4)}}{b^2 c^2 n^2 + B^4} = \mathit{ſin.}\,\lambda.$$

Coroll. III.

§. 576. Pro inveniendo angulo A ponatur ad abbreviandum $\mathit{ſin.}\,\beta = k$, $\mathit{ſin.}\,\lambda = m$, $\cos.\,\lambda = n$, $a^2 + b^2 \mp 2 a b n = h^2$, $\mathit{ſin.}\,(\beta + \psi) = x$, $\cos.\,(\beta + \psi) = \sqrt{(1 - x^2)}$, facta ſeparatione ſinuum ſubſtitutione et tranſpoſitione, habetur haec aequatio abbreviata: $h^2 k \mp (a - b n) c x = b c m \sqrt{(1 - x^2)}$, ex qua operatione abſoluta obtinetur: $x = \frac{\pm h k (a - b n) \pm b m \sqrt{(c^2 - h^2 k^2)}}{h c}$.

Scholion II.

§. 577. Omnium ſex caſuum in aequatione contentorum, ſunt tres Tetragonometriae proprii, qui tum obtinent, quando ex inventione ſingulorum trium in ſiniſtro triangulo figura conſtruenda proponitur, qui tres caſus conſtituunt totidem problemata nova in Geometria practica utilia et ſatis pulchra. Caetera tria utrique methodo et tetragonometricae et trigonometricae ſubſunt, ſed quorum ſolutiones tetragonometricae trigonometricis non adeo multum cedunt.

Coroll. IV.

§. 578. Si anguli A et B ponantur simul sumti aequales duobus rectis, ut prodeat trapezium parallelarum basium AD, BC, aequatio pro latere AD, adhibito signo anguli λ affirmativo, et angulo ψ eliminato in hanc mutatur:

$$c = \frac{(a^2+b^2 \mp 2ab \cos.\lambda) \sin.\beta}{\pm(a \sin.(\beta-\lambda) \pm b \sin.\beta)},$$ vel etiam angulo λ eliminato habetur:

$$c = \frac{(a^2+b^2 \pm 2ab \cos.\psi) \sin.\beta}{\pm(a \sin.(\beta+\psi) \mp b \sin.\beta)};$$ sed in altero casu, quo signum anguli λ negativum adhibetur erit angulo ψ eliminato:

$$c = \frac{a^2+b^2 \mp 2ab \cos.\lambda) \sin.\beta}{\pm(a \sin.(\beta-\lambda) \mp b \sin.(\beta-2\lambda)};$$ in aequatione pro latere AB, fit, $q=-k$, et aequatio in hanc paululum mutatur:

$$a = \frac{2bnk \pm cp \pm \surd((2bnk \pm cp))^2 - 4bk^2(b \mp c))}{2k}.$$

Pro latere BC fit etiam $q=-k$, et aequatio in hanc paululum mutatur:

$$b = \frac{2ank \pm ck \pm \surd((2ank \pm ck)^2 - 4ak(ak \pm cp))}{2k};$$

aequatio pro angulo ACD in hanc abit, adhibito signo anguli λ affirmativo, $tang.\ \beta = c(a \sin.\psi \mp b) : (a^2+b^2 \mp 2ab \cos.\lambda) + (a \cos.\psi \mp b)c)$. Verum signo anguli λ negativo usurpato habetur haec: $tang.\ \beta = c(a \sin.\psi \mp b \cos.2\lambda) : (a^2+b^2 \mp 2ab \cos.\lambda \mp c(a \cos.\psi \mp b \cos.2\lambda))$.

Coroll.

Coroll. V.

§. 579. Si angulus A ponatur rectus vel etiam aequalis tribus rectis, cadente vertice A supra diagonalem BD, et trapezio inverso prodeunte, aequatio pro latere AD in hanc mutatur:
$c = \frac{(a^2 + b^2 \mp 2ab\ cos.\lambda)\ sin.\beta}{\pm a\ cos.\beta \mp b\ cos.(\beta \pm \lambda)}$. In aequationibus pro lateribus AB et BC fit, $p = \pm cos.\beta$ $q = \pm cos.(\beta \pm \lambda)$, aequationes autem non mutantur, nisi quod situs signorum, quae terminis, in quibus occurrunt p et q, praeponuntur, ita mutatur, ut superius fiat inferius et contra. Aequatio pro angulo ACD in hanc mutatur: $tang.\ \beta = \pm (a - b\ sin.\lambda)\ c : (a^2 + b^2 \mp b\ (2a\ cos.\lambda \mp c\ sin.\lambda))$, denique in aequatione pro angulo B litterae p et q eosdem habent valores, signa vero termini primi in dextro aequationis membro respectu situs tantum mutantur.

Coroll. VI.

§. 540. Si angulus B sit rectus vel etiam aequalis tribus rectis, cadente vertice B ad dextram diagonalis AC, et trapezio inverso prodeunte, aequatio pro latere AD in hanc mutatur: $c = \frac{(a^2+b^2)\ sin.\beta}{\pm(a\ sin.(\beta+\psi) \pm b\ cos.(\beta+\psi))}$; in aequationibus pro lateribus caeteris fit, $n = o$, $q = \pm cos.(\beta+\psi)$, igitur aequatio pro latere AB in hanc mutatur:
$a = \frac{\pm cp \pm \sqrt{(c^2p^2 - 4bk(bk \pm cq))}}{2k}$, et aequatio

tio pro latere BC in hanc abit:

$$b = \frac{\mp cq \pm \surd(c^2q^2 - 4ak(ah^2 \pm cp))}{2k}.$$ At aequatio pro angulo ACD in hanc abit: $tang.\ \beta = (a \mp b)\ c\ fin.\ \psi : (a^2 + b^2 \mp (a\cos.\psi \mp b\,fin.\psi)\,c)$, denique in aequatione pro angulo A fit, $m = \pm 1$, $n = o$, $h^2 = a^2 + b^2$, ipsa vero aequatio in hanc mutatur valoribus restitutis: $$x = \frac{a\,fin.\beta \pm b\surd(c^2 - (a^2+b^2)fin.\beta^2)}{c\surd(a^2+b^2)}$$

Coroll. VII.

§. 581. Si angulus ACD ponatur rectus vel etiam aequalis tribus rectis, cadente latere CD supra vel infra latus BC, et trapezio inverso prodeunte, aequatio pro latere AD in hanc mutatur: $$c = \frac{a^2 + b^2 \mp 2ab\cos.\lambda}{\pm(a\cos.\psi \mp b\cos.(\psi \pm \lambda))};$$ in aequationibus pro lateribus caeteris fit, $k = \pm 1$, $p = \pm \cos.\beta$, $q = \pm \cos.(\psi \pm \lambda)$, aequatio igitur pro latere AB in hanc mutatur: $$a = \frac{2bn \pm cp \pm \surd(2bn \pm cp)^2 - 4b(b - cq))}{2},$$ aequatio pro latere BC in hanc mutatur: $$b = \frac{2an \pm cq \pm \surd(2an \mp cq)^2 - 4a(a - cp))}{2},$$

In aequatione pro angulo B fit, $A^2 = \pm (a^2 + b^2 - acp)$, $B^2 = \pm(2a - cp)\,b$, forma vero aequationis externa manet eadem, denique aequatio pro angulo A in hanc abit: $$x = \frac{\pm(a - bn)\,h \pm bm\surd(c^2 - h^2)}{hc} = \cos.\psi.$$

Coroll.

Coroll. VIII.

§. 582. Si ponatur angulus A aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus et trapezio in triangulum BCD abeunte, aequatio pro latere AD in hanc mutatur: $c = (a^2 + b^2 \mp 2ab \cos.\lambda)\, \sin.\beta : \mp (a \sin.\beta \pm b \sin.(\beta \pm \lambda))$; in aequationibus pro lateribus caeteris fit, $p = -k$, $q = -\sin.(\beta \pm \lambda)$, quarum igitur prior in hanc mutatur:

$$a = \frac{(2bn \mp c)k \pm \surd((2bn \mp c)^2 k^2 - 4bk(bk \mp cq}{2k},$$

et posterior in hanc abit:

$$b = \frac{2ank \pm cq \pm \surd((2ank \pm cq)^2 - 4ak^2(a \mp c))}{2k};$$

aequatio pro angulo ACD in hanc mutatur: $\tang.\beta = \pm cb\cos.\lambda : (a^2 + b^2 \mp 2ab\cos.\lambda \mp (b\cos.\lambda - a)c)$. Denique in aequatione pro angulo B fit, $A^2 = (a^2 + b^2 \pm ac)k$, $B^2 = bk(2a + c)$, quare in primo termino dextri membri signa tantum situm mutant.

Scholion III.

§. 583. Per hanc positionem solvuntur tria problemata nova ad triangula spectantia, quae vulgaribus methodis non solvuntur, qui casus tunc obtinent, quando quaeritur angulus B, latus BC, vel denique segmentum BE lateris BD, in singulo enim casu in triangulis partialibus non dantur nisi duo, nec in totali nisi duo, et angulus partialis ACD, vel etiam segmentum alterum ED, in eo casu quo quaeritur alterum EB, caeteri duo casus vulgaribus methodis etiam solvuntur.

Coroll.

Coroll. IX.

§. 584. Si ponatur angulus B aequalis duobus rectis lateribus BA, BC in diagonalem AC incidentibus, et trapezio in triangulum ACD abeunte; aequatio pro latere AD in hanc mutatur: $c=(a\pm b)^2 \, fin.\,\beta : (a\pm b)\, fin.\,(\beta+\psi)$, et consequenter, $c=(a\pm b)\, fin.\,\beta : fin.\,(\beta+\psi)$, quare signo negativo rejecto et solo adfirmativo retento, prodit solita analogia trigonometrica: $a+b : fin.(\beta+\psi) = c : fin.\,\beta$. In aequationibus pro segmentis AE et EC lateris AC, $n=-1$, $q=-fin.\,(\beta+\psi)$, quarum prior in hanc paululum mutatur:

$$a=\frac{-2bk\pm cp\pm\surd((2bk\pm cp)^2-4bk(bk\pm cq))}{2k},$$

posterior in hanc paululum transformatur:

$$b=\frac{-2ak\pm cq\pm\surd((2ak\mp cq)^2-4ak(ak\pm cp))}{2k};$$

aequatio pro angulo ACD in hanc mutatur:

$$tang.\,\beta=\frac{c\,(a\,fin.\,\psi\mp b\,cos.\,\psi)}{(a\mp b)^2\mp(a\pm b)\,c\,cos.\,\psi},$$ denique

in aequatione pro angulo A fit, $m=o$, $n=-1$, $h^2=(a\pm b)^2$, et aequatio ipsa in hanc mutatur:

$$x=\frac{(a+b)\,fin.\,\beta}{c}=fin.\,(\beta+\psi).$$

Scholion IV.

§. 585. Aequatio primo inventa valet in hypothesi figurae constructae. Scilicet quotiescunque

cunque trapezium eſt directum, ſi vero trapezium ita ſit inverſum ut latus AD cadat intra latus AB, valet haec aequatio: $(a^2+b^2 \mp 2ab\,cos.\lambda)$ $fin.\beta = \pm ac\,fin.(\beta-\psi) \mp bc\,fin.(\lambda \mp \beta \pm \psi)$, ubi notandum fieri etiam poſſe, ut ſit angulus ψ major angulo β, in quo caſu rectius ſcriberetur aequatio hoc modo: $(a^2+b^2 \mp 2ab\ cos.\ \lambda)$ $fin.\beta = \pm ac\,fin.(\psi-\beta) \mp bc\,fin.(\lambda \mp \beta \pm \psi)$, quod ſi vero latus AD cadat extra latus AB, quomodocunque trapezium ſit inverſum, valet aequatio primo inventa. Aequatio igitur generaliſſima, hoc problema reſolvens eſt hujus modi:

$$(a^2+b^2 \mp 2ab\,cos.\,d)\,fin.\beta = \pm ac\,fin.\begin{pmatrix}\beta \pm \psi \\ \psi - \beta\end{pmatrix}$$

$$\mp bc\,fin.\begin{pmatrix}\beta + \psi \mp \lambda \\ \lambda \mp \beta \pm \psi\end{pmatrix}.$$

Problema XLII.

§. 586. In figura quadrilatera propoſita $ABCD$, inter haec ſex: latus $AB=a$, $BC=b$, $DC=d$, angulum $A=\psi$, $B=\lambda$, et $ACD=\beta$; aequationem invenire. Fig. XLII.

Solutio.

Cum ſit per antecedens problema diagonalis $AC = h$, et

$$fin.CAD = \frac{fin.\psi \surd(h^2-b^2 fin.\lambda^2) - b\,fin.\lambda\,cos.\psi}{h},$$

nec non $fin.\ D = \pm\ a\ fin.\ (\beta+\psi) \mp b$ $fin.\ (\beta+\psi \pm \lambda)$, in triangulo dextro pervenio ſtatim ad hanc analogiam: $AC : fin.D = CD :$ *fin.*

fin. CAD, hoc eſt ſymbolis ſubſtitutis:
$h : \frac{\pm a\, fin.\, (\beta+\psi) \mp b\, fin.\, (\beta+\psi \pm \lambda)}{h} =$

$= d : \frac{fin.\, \psi\, (a - b\, cos.\, \lambda) - b\, fin.\, \lambda\, cos.\, \psi}{h}$

conſequenter habetur haec aequatio:
$\pm a\, d\, fin.\, (\beta+\psi) \mp b\, d\, fin.\, (\beta+\psi \pm \lambda) =$
$= (\pm a\, fin.\, \psi \mp b\, fin.\, (\lambda \pm \psi)$
$\surd\, (a^2 + b^2 \mp 2\, a\, b\, cos.\, \lambda)$, Q. E. I.

Scholion I.

§. 587. De hac aequatione monendum, quod obtinente ſigno affirmativo termini ſiniſtimi in utroque aequationis membro, obtinere poſſe utrumque ſignum termini $b\, cos.\, \lambda$. Sed obtinente ſigno negativo ejusdem ſiniſtimi poſſe tantum obtinere ſignum negativum ejusdem termini $b\, cos.\, \lambda$, cum fieri non poſſit, ut in eodem triangulo rectilineo duo anguli exiſtant obtuſi.

Coroll. I.

§. 588. Ex hac aequatione ſtatim obtinetur: latus $CD = d =$

$$= \frac{(\pm a\, fin.\, \psi \mp b\, fin.\, (\beta+\psi \pm \lambda)\, \surd\, (a^2+b^2 \pm 2ab\, cos.\, \lambda)}{\pm a\, fin.\, (\beta+\psi) \mp b\, fin.\, (\beta+\psi \pm \lambda)}$$

Ad obtinendum latus AB ſit ad abbreviandum *fin.* $\psi = k$, *fin.* $(\lambda \pm \psi) = m$, *fin.* $(\beta+\psi) = n$, *cos.* $\lambda = q$, *fin.* $(\beta+\psi \pm \lambda) = p$, erit facta ſubſtitutione aequatio abbreviata: $\pm\, a\, d\, n \mp b\, d\, p = (a\, k \mp b\, m)\, \surd\, (a^2 + b^2 \mp 2\, a\, b\, q)$, cujus ſumtis qua-

quadratis, facta transpositione et ordinatione, obtinetur haec aequatio quarti gradus:

$$a^4 \pm \frac{2ab(m+kq)}{k} + \frac{a^2}{k^2}(m^2+k^2 \pm 4mkq)b^2-d^2n^2)$$

$$\pm \frac{2ab}{k^2}(mk+m^2q)b^2+d^2np) + \frac{b^2}{k^2}(b^2m^2-d^2p^2 = 0.$$

Sed pro latere BC habetur haec fere similis quarti gradus: $b^4 \pm \frac{2.ab^3}{m}(k+mq)$

$$+b^2(+\frac{k^2 1}{m^2} \pm \frac{4kq}{m})a^2-d^2p^2) \pm b(2a^3k(\frac{1}{m}+\frac{kq}{m^2})$$

$$+2ad^2np) + a^2(a^2-\frac{d^2n^2)}{m^2} = 0.$$

Coroll. II.

§. 589. Pro angulo A obtinendo, finibus a se invicem separatis, et posito ad abbreviandum: $\sqrt{}(a^2+b^2 \mp 2ab\ cos.\lambda) = h$, $fin.\beta = m$, $cos.\beta = n$, $fin.(\beta \pm \lambda) = p$, $cos.(\beta \pm \lambda) = q$, $fin.\lambda = r$, $cos.\lambda = s$, facta substitutione et ad finem perducta operatione, pervenietur ad hanc aequationem: $tang.\psi = \frac{\pm(br-dp)b-adm}{(an-dq)d-(a-b-1)h}$; ad angulum ACD inveniendum posito ad abbreviandum toto dextro aequationis membro $= M^2$, et ut ante $fin.\lambda = r$, $cos.\lambda = s$, $fin.(\beta+\psi) = x$, $cos.(\beta+\psi) = \sqrt{}(1-x)^2$ facta substitutione et transpositione erit aequatio abbreviata: $\pm(a-bs)dx - M^2 = \mp bdr\sqrt{}(1-x^2)$, sumtis quadratis,

facta concinnatione et posito ut supra $a^2 + b^2 \mp 2abs = h^2$, operatione ad finem perducta, pervenitur ad hanc aequationem:

$$x = \frac{\pm M^2(a-bs) \pm br\sqrt{(d^2h^2 - M^4)}}{dh^2} = \sin.(\beta + \psi),$$

et consequenter ad ipsum angulum descendendo habetur:

$$\beta = ang. \sin. \frac{\pm M^2(a-bs) \pm br\sqrt{(d^2h^2 - M^4)}}{dh^2} - \psi.$$

Scholion II.

§. 590. Inter sex casus, qui in aequatione continentur, tres sunt Tetragonometriae proprii, qui tum obtinent, quando ea, quae in sinistro triangulo designata sunt, quaeruntur, quorum casuum duorum solutiones perduxi ad aequationes quarti gradus, quando quaeritur latus AB vel BC; tertii casus evolutionem, quando angulus B quaeritur, consulto omisi, cum aequatio ad sextum gradum ascendat et ingenti constet multitudine terminorum, et consequenter parum sit utilis. Tres autem hi casus constituunt in Geometria practica totidem problemata nova, utilia, et satis etiam pulchra. Caeteri tres casus utrique methodo et tetragonometricae et trigonometricae subsunt; sed itá tamen ut trigonometrice fere facilius et expeditius solvantur, quos tamen casus evolvi quia ex aequatione haud operose sequntur.

Coroll.

Coroll. III.

§. 591. Si anguli A et B ponantur ſimul ſumti duobus rectis aequales, prodeunte trapezio parallelarum baſium AD, BC, aequatio pro latere CD in caſu anguli λ affirmativi in hanc mutatur:

$$d = \frac{\pm(a\,fin.\,\psi + b\,fin.\,\beta)\sqrt{(a^2+b^2 \mp 2ab\,cos.\,d)}}{a\,fin.(\beta+\psi)+b\,fin.\,\beta}.$$

Sed in caſu anguli λ negative ſumti habetur haec altera:

$$d = \frac{\pm(a\,fin.\,\psi + b\,fin.(\beta-2\lambda))\sqrt{(a^2+b^2 \mp 2ab\,cos.\,d)}}{a\,fin.(\beta+\psi)+b\,fin.(\beta-2\lambda)}.$$

In aequationibus pro lateribus caeteris fit in eo caſu, quo adhibetur ſignum anguli ψ affirmativum, $m=o$, et $p=-fin.\,\beta$, quare aequatio pro latere AB in hanc mutatur et contrahitur:

$$a^4 \pm 2a^3bq + a^2\frac{(b^2-d^2n^2)}{k} \pm \frac{2abd^2np}{k^2} - \frac{b^2d^2p^2}{k^2} = o.$$

Sed aequatio pro latere BC ad ſecundum gradum deſcendendo in hanc mutatur et contrahitur: $b^2(a^2k^2-d^2p^2) \pm 2ab(a^2k^2q-d^2np) + a^2(a^2k^2-d^2n^2) = o$, ex qua operatione abſoluta elicitur: $b = \frac{\mp a(a^2k^2q - d^2np) \pm a^2k}{a^2k^2-d^2p^2}$

$\frac{\sqrt{(a^2k^2(q-1)-d^2(p^2+n^2-2npq))}}{a^2k^2-d^2p^2}$; in altero

 caſu,

casu, quo adhibetur signum negativum, fit $m = \text{sin.}\, 2\lambda$, $p = \text{sin.}\,(\beta - 2\lambda)$; caeterum aequationes quoad formam externam manent invariatae.

Coroll. IV.

§. 592. Si angulus A sit rectus vel etiam aequalis tribus rectis, cadente vertice A supra diagonalem BD, et trapezio inverso prodeunte, aequatio pro latere AB in hanc mutatur:

$$d = \frac{(\pm a \mp b\, \text{cos.}(\beta \pm \lambda)) \surd (a^2 + b^2 \pm 2ab\, \text{cos.}\lambda)}{a\, \text{cos.}\beta - b\, \text{cos.}(\beta \pm \lambda)},$$

vel etiam ita,

$$d = \frac{\pm(a - b\, \text{cos.}(\beta \pm \lambda)) \surd (a^2 + b^2 \mp 2ab\, \text{cos.}\lambda)}{a\, \text{cos.}\beta - b\, \text{cos.}(\beta \pm \lambda)};$$

In aequationibus pro lateribus caeteris fit, $k = \pm 1$, $m = \pm\, \text{cos.}\lambda = q$, $n = \pm\, \text{cos.}\beta$, $p = \pm\, \text{cos.}(\beta \pm \lambda)$, quarum itaque prior in hanc mutatur:

$$a^4 \pm 4a^3bm + a^2(1 + m^2 \pm 4m^2)b^2 - d^2n^2)$$
$$\pm 2ab((1+m)b^2m^2 + d^2np) + b^2(b^2m^2 -)d^2p^2) = 0,$$

Sed in hanc posterior:

$$b^4 + \frac{2ab^3}{m}(1 + m^2) + \frac{b^2}{m^2}\frac{(1 + m^2 \pm 4m^2)}{m^2}a^2 - d^2p^2)$$
$$\pm 2ab((1 \pm 1)\frac{a^2}{m} + d^2np) + a^2\frac{(a^2 - d^2n^2)}{m^2} = 0;$$

In aequatione pro angulo ACD fit,

$$M^2 = \pm(a - b\, \text{cos.}\lambda) \surd (a^2 + b^2 \pm 2ab\, \text{cos.}\lambda)$$

et

et $x = \pm \cos.\beta$, et aequatio in hanc paululum mutatur:

$$\cos.\beta = \frac{M^2(a-bs) \pm br \surd (d^2h^2 - M^4)}{dh^2},$$

et ipfe angulus:

$$\beta = ang.\ cos. \frac{M^2(a^2-bs) \pm br \surd (d^2h^2 - M^4)}{dh^2}.$$

Coroll. V.

§. 593. Si angulus B ponatur rectus vel etiam aequalis tribus rectis, cadente vertice B ad dextram diagonalis AC, et trapezio inverfo prodeunte, aequatio pro latere DC in hanc mutatur:

$$d = \frac{(\pm a\ fin.\psi \mp b\ cos.(\beta+\psi)) \surd (a^2+b^2)}{\pm a\ fin.(\beta+\psi) \mp b\ cos.(\beta+\psi)},$$

vel etiam:

$$d = \frac{\pm (a\ fin.\psi - b\ cos.(\beta+\psi)) \surd (a^2+b^2)}{a\ fin.(\beta+\psi) - b\ cos.(\beta+\psi)}.$$

Sed in aequationibus pro lateribus caeteris fit, $m = \pm cos.\psi$, $q = o$, $p = \pm cos.(\beta+\psi)$, quarum itaque prior in hanc mutatur: $a^4 \pm 2ad^3b\ cot.\psi + a^2(b^2(1+cot.\psi^2) - d^2n^2) \pm 2a b (b^2 cot.\psi + d^2 n p) + b^2 (b^2 cot.\psi^2 - d^2p^2)$ $\overline{\quad h^2 \quad} = o$,

eft pofterior in hanc abit: $b^4 \pm 2ab^3 tang.\psi + b^2 (a^2 fec.\psi^2 - d^2p^2) \pm 2ab(a^2 tang\psi + d^2 n p) + a^2(a^2 - d^2n^2)$ $\overline{\quad m^2 \quad} = o$. In aequatione pro an-

gulo A fit, $h = \sqrt{(a^2+b^2)}$, $p = \pm \cos. \beta$, $q = \mp \sin. \beta$, $r = \pm 1$, $s = 0$, quae igitur in hanc mutatur:

$$tang. \psi = \frac{\pm (b - d p)' - a'd m}{(a n \pm d q) - a h};$$

denique in aequatione pro angulo ACD fit, $M^2 = (\pm a \sin. \psi \mp b \cos. \psi) \sqrt{(a^2 + b^2)}$ $h^2 = (a^2 + b^2)$, quae igitur in hanc mutatur:

$$\sin. (\beta + \psi) = \frac{\pm a M^2 \pm b \sqrt{(d^2 h^2 - M^4)}}{d h^2},$$

et consequenter ipse angulus:

$$\beta = ang. \sin. \frac{\pm a M^2 \pm b \sqrt{(d^2 h^2 - M^4}}{d h^2} - \psi.$$

Coroll. VI.

§. 594. Si angulus ACD fit rectus vel etiam tribus rectis aequalis, cadente latere CD supra vel infra latus CB, et trapezio inverso prodeunte, aequatio pro latere CD in hanc abit:

$$d = \frac{\pm (a \sin. \psi - b \cos. (\psi \mp \lambda))}{a \cos. \psi - b \cos. (\psi \pm \lambda)}$$

$$\frac{\sqrt{(a^2 + b^2 \pm 2 a b \cos. \lambda)}}{a \cos. \psi - b \cos. (\psi \pm \lambda)}.$$ In aequationibus pro lateribus caeteris fit, $n = \pm \cos. \psi$, $p = \pm \cos. (\psi \pm \lambda)$, sed aequationum forma externa invariata manet. In aequatione pro angulo

angulo A fit, $m = \pm 1$, $n = o$, $p = \pm \cos. \lambda = \pm s$, $q = \mp \sin. \lambda = \mp r$, ipſa igitur aequatio in hanc mutatur:

$$tang. \psi = \frac{(b\,r \pm d\,p)\; b \mp a\,d)}{(a - b\,s)\; h \pm d^2\,q}.$$

Coroll. VII.

§. 595. Quod ſi angulus A ponatur aequalis duobus rectis, lateribus AB, AD in diagonalem BD incidentibus et trapezio in triangulum BCD abeunte, aequatio pro latere CD, in hanc abit:

$$d = \frac{\pm\, b\, \sin. (\beta \pm \lambda)\; \surd (a^2 + b^2 \mp 2\,a\,b\, \cos. \lambda)}{b\, \sin. (\beta \pm \lambda) - a\, \sin. \beta}.$$

In aequationibus pro lateribus caeteris erit $k = o$, $m = \mp \sin. \lambda$, $n = - \sin. \beta$, $p = \mp \sin. (\beta \pm \lambda)$, aequatio igitur pro ſegmento BE lateris BD ita mutatur ut extracta radice prodeat haec aequatio quadratica: $a = \frac{b((b^2 m^2 q + d^2 n p) \pm b\,m}{b^2\,m^2 - d^2\,n^2}$

$$\frac{\surd\, (b^2\,m^2\,(q - 1) + d^2\,(p^2 \pm 2\,n\,p\,q + n^2)))}{b^2\,m^2 - d^2\,n^2}.$$

Verum aequatio pro latere BC in hanc mutatur: $b^4 \pm 2\,a\,b^3\,q + b^2\,(a^2 - d^2\,p^2 \mp \frac{2\,a\,b\,d^2\,n\,p}{m}$

$+ \frac{a^2\,(a^2 - d^2 n^2)}{m^2} = o.$ Denique in aequatione pro angulo $A\,C\,D$, fit $M^2 = b\, \sin. \lambda\; \surd\, (a^2 + b^2 \mp 2\,a\,b\, \cos. \lambda)$, et $x = - \sin. \beta$,

 quae

quae igitur valore ipſius h reſtituto, qui in M^2 continetur, fit talis:

$$fin.\ \beta = \frac{\mp br - ((a-bs) \pm \sqrt{(d^2 - b^2 r^2)})}{dh}.$$

Scholion III.

§. 596. Per poſitionem in hoc Corollario adhibitam ſolvitur unum ad triangula ſpectans problema, quod tum obtinet, cum latus BC quaeritur, nam in triangulis partialibus non habentur niſi duo data, et in triangulo totali habentur etiam duo et praeterea ſegmentum AB, et angulus partialis ACD. Praeter hunc caſum etiam alter daretur, quando quaeritur angulus B, ſiquidem aequatio et ſolutio aliqua utilis tetragonometrica hujus caſus generalis haberetur. Reliqui caſus licet tetragonometrice ſolvantur, tamen trigonometrice facilius expediuntur.

Scholion IV.

§. 597. Quod ſi latus AB cadat extra latus AD, quomodocunque trapezium ſit inverſum, valet haec aequatio: $ad\ fin.(\beta - \psi \mp b\ fin.(\beta - \psi \mp \lambda) = (b\ fin.(\lambda \pm \psi) \mp a\ fin.\psi)\ \sqrt{(a^2 + b^2 \mp 2ab\ cos.\lambda)}$. Si latus AB cadat intra latus AD, quomodocunque trapezium ſit inverſum valet haec aequatio: $\pm ad\ fin.(\beta + \psi) \mp bd\ fin.(\beta + \psi \mp \lambda) = (b\ fin.(\lambda \mp \psi) \pm a\ fin.\psi)\ \sqrt{(a^2 + b^2 \mp 2ab\ cos.\lambda)}$. Sed pro trapezio

pezio directo inventa fuit haec aequatio: $\mp a\, d\, fin.\, (\beta + \psi) \mp b\, d\, fin.\, (\beta + \psi \pm \lambda) = = (\pm a\, fin.\, \psi \mp b\, fin.\, (\lambda \pm \psi))\, \sqrt{} (a^2 + b^2 \mp 2\, a\, b\, cos.\, \lambda)$. Quare pro omni trapezio haec est generalis aequatio: $\pm a\, d\, fin.\, (\beta \pm \psi) \mp b\, d\, fin.\, (\beta \pm \psi \pm \lambda) = = (\pm a\, fin.\, \psi \mp b\, fin.\, (\lambda \pm \psi)\, \sqrt{} (a^2 + b^2 \mp 2\, a\, b\, cos.\, \lambda)$.

Scholion. V.

§. 598. Fuit hoc problema quadragesimum secundum et quidem ex recensione et determinatione Domini Lambert omnium ultimum; verum supra Cap. II. (§. 21.) dixi, videri posse, omissum esse casum illum, qui prodiret, si manentibus, ut in hoc ultimo problemate omnibus determinationibus in triangulo sinistro, loco lateris in dextro triangulo, adhiberetur angulus *D*, qui casus an recte an secus a Clariss. Lambent sit praetermissus jam dispiciam. In hoc itaque casu, cum sit pro trapezio directo, $fin.\, C\, A\, B = - fin.\, (\beta + \varphi + \psi)$ et $fin.\, A\, C\, B = - fin.\, (\beta + \lambda + \varphi + \psi)$, formata in triangulo sinistro hac analogia: $AB : fin.\, A\, C\, B = B\, C : fin.\, C\, A\, B$, h. e. $a : - fin.\, (\beta + \lambda + \varphi + \psi) = b : - fin.\, (\beta + \varphi + \psi)$, et consequenter habetur haec aequatio: $a\, fin.\, (\beta + \varphi + \psi) = b\, fin.\, (\beta + \lambda + \varphi + \psi)$; quantum igitur ad aequationis formam, quae a praecedentibus omnibus differt, hic casus minus recte praetermissus videtur, nec ex ea ratione praetermitti debuit, quod aequatio nullum contineat casum Tetrago-

nometriae proprium, tales enim etiam exstiterunt aliquot aequationes supra allatae, non tamen Clariss. Lambert illa problemata ex Tetragonometria rejecit, quare neque hoc problema ex hac ratione ex Tetragonometria eliminari oportuit.

Coroll. VIII.

§. 599. Ex hac aequatione est:

$$a = \frac{b \; sin. \; (\beta + \lambda + \varphi + \psi)}{sin. \; (\beta + \varphi + \psi)}$$ et

$$b = \frac{a \; sin. \; (\beta + \varphi + \psi)}{sin. \; (\beta + \lambda + \varphi + \psi)}$$; pro obtinendo angulo A habetur haec aequatio:

$$tang. \psi = \frac{b \; sin. \; (\lambda + \varphi + \beta) - a \; sin. \; (\beta + \varphi)}{a \; cos. \; (\beta + \varphi) - b \; cos. \; (\lambda + \beta + \psi)};$$

pro angulo D habetur haec aequatio:

$$tang. \varphi = \frac{b \; sin. (\lambda + \beta + \psi) - a \; sin. (\beta + \psi)}{a \; cos. \; (\beta + \psi) - b \; cos. (\beta + \lambda + \psi)}$$

pro angulo B obtinendo, cum sit

$$sin. (\beta + \lambda + \varphi + \psi) = \frac{a \; sin. \; (\beta + \varphi + \psi)}{b},$$ sequitur esse

$$\lambda = ang. \; sin. \frac{a \; sin. (\beta + \varphi + \psi)}{b} - \beta - \varphi - \psi.$$ Ad angulum ACD obtinendum invenitur esse:

$$tang. \; (\beta + \varphi + \psi) = \frac{b \; sin. \; \lambda}{a - b \; cos. \; \lambda},$$ et consequenter habetur ipse angulus:

$$\beta = ang. \; tang. \frac{b \; sin. \; \lambda}{a - b \; cos. \; \lambda} - \varphi - \psi.$$

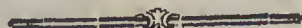
F I N I S.

MONITUM.

Mſcr. Lipſiam Hafnia exmiſſum, ad cujus legem hoc Tetragonometriae Opus excuſum litteris, correctumque prodiit, nitore ſcriptionis ſe valde commendavit, quin et hinc inde commiſſa feſtinanti forte calamo ſphalmata, penicillo emendata reſtituit; quare factum eſt, ut et Typotheta liberius formas litterarum colligere poſſet, nec Correctori relinqueret admodum multa, quae vel deleret, vel omiſſa etiam reſtitueret. Haec tamen eo non ſpectant, ut credant Lectores, adeo nihil erratum eſſe prorſus, quam quidem ſpem tale calculi genus aegre admittere videtur. Ea, quae repetita lectio exhibuit, adeo non morantur quemquam (et rariora haec errata ſunt) ut vel ob id ipſum non notarentur in calce operis. Quod ad ſigna : et ? adtinet, datum fuit nonnihil Grammaticae, et conſuetudini inter nos receptis. Itaque lector quisquis fuerit, facile valorem grammaticum diſtinxerit ab uſu, quem prius ſignum in analogiis geometricis obtinet, neque profecto alterum in calce lectorum Problematum deſideraverit; quamvis negari nequeat, ſubtilem quandam quaſi interrogandi rationem in omni Problemate ineſſe.

Tandem tardius aliquantum intelleximus, modum ſignandi litteras formularum algebraicarum p. videlicet atque q. quo arbitrario uſus fuit Auctor non innuere, ſumendas eſſe ex alio formae arctioris genere; itaque cum in exemplari ſcripto videremus, nudam litteram p. deletam, eique in margine ſubſtitutam fuiſſe hoc modo ſignatam: p, cujus ſigni rationem

tionem nullibi explicatam deprehendere licuit: facile fuit colligere, p, esse debere formae diversae, itaque virgulam infra caput litterae horizontalem negligendam esse. Factum hoc alibi non reperias, quam pag. 134. lin. ultim. in qua pro p. poni debuisset p quemadmodum videre est in formula algebraica p. 165. §. 238. Nec alibi facile quantum nunc quidem recordamur, occurrunt valores quaesitorum tetragonometricorum determinati calculo, litteris p vel etiam q distincti. Haec erant, de quibus benevolos Lectores monere e re videbatur; ad Typum vero, quod adtinet, certo persuasi sumus eum sese propria elegantia abunde comprobaturum.



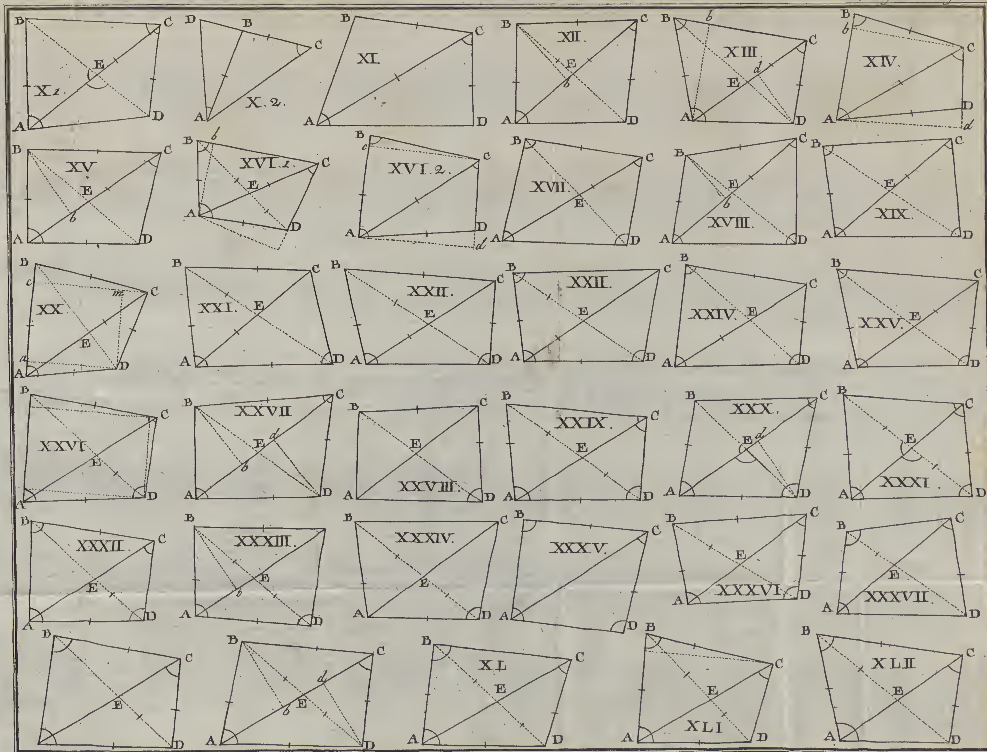

X 1.
X. 2.
XI.
XII.
XIII.
XIV.
XV.
XVI. 1.
XVI. 2.
XVII.
XVIII.
XIX.
XX.
XXI.
XXII.
XXIII.
XXIV.
XXV.
XXVI.
XXVII.
XXVIII.
XXIX.
XXX.
XXXI.
XXXII.
XXXIII.
XXXIV.
XXXV.
XXXVI.
XXXVII.
XL.
XLI.
XLII.

Fig. XLII.



